Attaching Galois Representations to Automorphic Representations

February 28, 2023

The aim of this talk is to establish our first connection between the Hecke rings of Arun's talk, and the Galois deformation rings of Zach and James' talks. This goes via the construction of Galois representations from automorphic forms. First, we discuss the history and why these were originally constructed. Then we move on to sketch the original constructions of Eichler, Kuga-Sato, Shimura. Then the fundamental work of Deligne, which forms the framework for the general case. Finally we state what is known in general.

Throughout, we ignore issues at the cusps (i.e. cohomology is actually interior cohomology etc).

1 Why Galois Representations were Attached to Modular Forms

Let

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

be the discriminant modular form (of weight 12 and level $\mathrm{SL}_2(\mathbb{Z})$), and the Ramanujan τ-function. In the original work of Ramanujan on the τ function, he made two conjectures (as well as many others): for each prime p,

1. $|\tau(p)| \leq 2p^{11/2}$; and,
2. $\tau(p) \equiv 1 + p^{11} \pmod{691}$.

The first remained for many years, the well known Ramanujan Conjecture. The second Ramanujan himself proved, however it remained a mystery how these sorts of congruences systematically arose.

For the first conjecture, Weil made an important observation that the bound is very similar to the bounds that he had proven for exponential sums. Work of Hasse and Davenport [HD] had related exponential sums to point counts of curves over finite fields. That is, to bound the sum

$$S_{f,\lambda} = \sum_{a \pmod{p}} e^{\lambda \frac{2\pi i f(a)}{p}}$$

where $f(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree d, they considered the Artin-Schreier curve

$$X : y^p - y = f(x)$$

which has genus $\frac{1}{2}(p-1)(d-1)$ and showed that

$$L_{X/\mathbb{F}_p}(t) = \prod_{\substack{1 \leq i \leq d-1 \\ 1 \leq \lambda \leq p-1}} (1 - \omega_{i,\lambda} t)$$

where $\omega_{i,j} \in \mathbb{C}$ satisfy

$$\sum_{1 \leq i \leq d-1} \omega_{i,\lambda} = S_{f,\lambda}.$$

In particular this means that

$$|X(\mathbb{F}_p)| - p - 1 = \sum_{1 \le \lambda \le p-1} S_{f,\lambda}.$$

The Riemann hypothesis of Artin for curves over finite fields, which has already been proved by Hasse in the case of elliptic curves, conjectures that $|\omega_i| = p^{1/2}$. Weil realised bounding the exponential sums therefore followed from the Riemann hypothesis for arbitrary algebraic curves, which he proved. This given the bound

$$|S_{f,\lambda}| \le (d-1)p^{1/2}.$$

Weil realised that the exponent is related to the degree of cohomology (and the constant factor is related to the dimension of the corresponding part of cohomology - i.e. the motive) and calculations of this sort eventually led to his statement of the Weil conjectures. He understood that the conjecture of Ramanujan could be settled in this cohomological fashion if the coefficients of the τ-function could be realised inside the point counts of some algebraic variety over a finite field, corresponding to some two dimensional piece of the degree 11 cohomology. At the time, no such sensible cohomology theory existed.

Meanwhile, a relatively separate strand of mathematics was growing in discovering and proving more congruences of the τ-function, and coefficients of other modular forms. Although there seems to have been quite an industry in this, there was no general framework for studying congruences, or understanding in which cases no congruences existed. Serre came to study this in [Ser67] after some time spent studying the l-adic representations attached to elliptic curves (the Tate modules), where similar congruence phenomena arose and could be understood via the Galois representations.

For example, consider the elliptic curve 26.b2 which has conductor 26 and Weierstrass equation

$$E : y^2 + xy + y = x^3 - x^2 - 3x + 3.$$

This has a torsion point $(1, 0)$ of order 7. Therefore, it's 7-torsion as a Galois representation has a fixed vector, and cyclotomic determinant (as always), and so is equivalent to a representation

$$\begin{pmatrix} 1 & * \\ 0 & \epsilon \end{pmatrix}.$$

Therefore for any $l \nmid 7 * 26$,

$$a_l = \mathrm{Tr}(\mathrm{Frob}_l) \equiv 1 + l \pmod 7.$$

Indeed,

$$a_3 = -3, a_5 = -1, a_{11} = -2, ...$$

This congruence can be rephrased as the fact that the existence of torsion implies that the image of

$$\rho_{E,7} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{Z}_7)$$

has image in

$$\left\{ g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod 7 \right\}.$$

The study of the Tate modules, and the developing theory of etale cohomology relating the l-adic representations to cohomology was the final missing piece - Serre realised that Weil's dream of realising the τ-function via Frobenius on cohomology was compatible with the study of congruences of the τ-function: both were to be realised in an attached l-adic representation which should come somehow from cohomology!

Thus, Serre's conjecture was born:

Conjecture 1.1. There should exist a continuous representation

$$\rho : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{Q}_p)$$

which satisfies

- it is unramified away from p;
- for $l \neq p$,

$$\det(1 - \mathrm{Frob}_p X) = 1 - \tau(p)X + p^{11}X^2$$

Furthermore, this should be a subquotient of étale cohomology in degree 11 of some algebraic variety over $\mathbb{Q}$, in which case the Weil conjectures imply the Ramanujan conjecture.

Clearly, it would be possible and interesting to generalise this the all modular forms.

Theorem 1.2. Let $N \geq 1$ and $k \geq 2$ be integers and $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character. Let $f = \sum_n a_n q^n \in S_k(N, \epsilon)$ be a normalized eigenform and $\lambda | l$ be a place of $\mathbb{Q}(f)$. Then, there exists an l-adic representation $V_{f,\Lambda}$ over $\mathbb{Q}(f)_\lambda$ such that for every $p \nmid Nl$, $V_{f,\lambda}$ is unramified at p and

$$\mathrm{Tr}(\phi_p | V_{f,\lambda}) = a_p(f).$$

This simultaneously gives a common framework for both of the previous lines of thought:

- If this l-adic representation comes from cohomology in degree 11, and the Weil conjectures are true, then get the correct bound on $\tau(p)$ as it is the sum of two Weil p-numbers of weight 11.
- Suppose we consider the mod 691 representation and we can show that there is a fixed vector, then the representation is equivalent to one of the form

$$\begin{pmatrix} 1 & * \\ 0 & \epsilon^{11} \end{pmatrix}$$

  where ϵ is the mod 691 cyclotomic character. This means therefore that the trace is

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

  Therefore congruences that had been observed before we perfectly explained by understanding the image of the mod l representations.

Therefore we are certainly interested in constructing l-adic Galois representations attached to modular forms. We saw that modular forms can be related to certain cohomology groups last time via the Eichler-Shimura isomorphism. These were sheaf cohomology groups on the complex analytic modular curves, so there is no Galois representation. Therefore we should aim to construct the associated l-adic representation. The general conjecture was proven by Deligne in [**Deligne1971**] via this strategy, which we will sketch now.

2 Modular Curves and Hecke Operators

For an integer $N \geq 3$, there is a functor

$$\begin{aligned} F_N : \underline{\mathrm{Sch}} &\longrightarrow \underline{\mathrm{Set}} \\ S &\longmapsto \left\{ \text{pairs } (E, \alpha) \,\middle|\, \begin{array}{r} \text{elliptic scheme } E \to S \\ \alpha : \mathbb{Z}/N \hookrightarrow E[N] \end{array} \right\} \Big/ \sim \end{aligned}$$

which is representable by a (disconnected) scheme $Y_{\Gamma_1(N)}$ over $\mathrm{Spec}\,(\mathbb{Z}\,[1/N])$. (Note the slight change of notation between mine and Arun's talks - for me, modular curves will be the full disconnected versions). This comes with a universal elliptic curve

$$\mathcal{E} \to Y_{\Gamma_1(N)}.$$

We need another modular curve defined by a moduli functor that has not been mentioned yet. Let p be a prime not dividing N. Then we define a functor $F_{N,p}$ which assigns to each scheme S the isomorphism classes of commutative diagrams of S-schemes

$$\begin{aligned} F_{N,p} : \underline{\mathrm{Sch}} &\longrightarrow \underline{\mathrm{Set}} \\ S &\longmapsto \left\{\text{diagrams } \mathcal{D} = \mathcal{D}(E_1, E_2, \phi, \alpha_1, \alpha_2)_{/S}\right\} / \sim \end{aligned}$$

where $\mathcal{D} = \mathcal{D}(E_1, E_2, \phi, \alpha_1, \alpha_2)_{/S}$ denotes the following commutative diagram of S-schemes

$$\begin{array}{ccccc} & & \mathbb{Z}/N & & \\ & \swarrow^{\alpha_1} & & \searrow^{\alpha_2} & \\ E_1[N] & & \longrightarrow & & E_2[N] \\ \downarrow & & & & \downarrow \\ E_1 & & \xrightarrow{\ \phi\ } & & E_2 \end{array}$$

where E_1, E_2 are elliptic schemes over S, ϕ is a p-isogeny (i.e. a homomorphism of group schemes whose kernel is a finite group scheme of order p, note that the kernel could have only one point, even at a geometric point of S, for example the kernel of Frobenius at any characteristic p point), and α_1, α_2 are isomorphisms. The functor $F_{N,p}$ is represented by a modular curve $Y_{\Gamma_1(N;p)}$ where $\Gamma(N;p) := \Gamma_1(N) \cap \Gamma_0(p)$. There are two natural transformations

$$\begin{aligned} g_1, g_2 : F_{N,p} &\longrightarrow F_N \\ g_1 : \mathcal{D} &\longmapsto (E_1, \alpha_1) \\ g_2 : \mathcal{D} &\longmapsto (E_2, \alpha_2) \end{aligned}$$

given by taking just the left or right hand sides of this diagram. These induce (via Yoneda) natural maps

$$g_1, g_2 : Y_{\Gamma_1(N;p)} \to Y_{\Gamma_1(N)}.$$

There is also a natural involution

$$\begin{aligned} \sigma : F_{N,p} &\longrightarrow F_{N,p} \\ \mathcal{D} &\longmapsto (E_2, E_1, \alpha_2, p\alpha_1, \widehat{\phi}) \end{aligned}$$

of $F_{N,p}$ by taking the transpose of the horizontal maps.

Lemma 2.1. After base changing to $\mathbb{C}$ and taking analytification, there is an isomorphism (on each component) between the two correspondences

$$\begin{array}{ccc} Y_{\Gamma_1(N;p)}(\mathbb{C})^{an} & \xrightarrow{\ \sigma\ } & Y_{\Gamma_1(N;p)}(\mathbb{C})^{an} \\ \downarrow g_1 & \searrow^{g_2} & \downarrow g_1 \\ Y_{\Gamma_1(N)}(\mathbb{C})^{an} & & Y_{\Gamma_1(N)}(\mathbb{C})^{an} \end{array} \qquad \begin{array}{ccc} \Gamma_{\alpha_p} \setminus \mathfrak{H}^* & \xrightarrow{\ [\alpha_p]\ } & \Gamma^{\alpha_p} \setminus \mathfrak{H}^* \\ \downarrow & & \downarrow \\ \Gamma \setminus \mathfrak{H}^* & & \gamma \setminus \mathfrak{H}^* \end{array}$$

where $\alpha_p := \begin{pmatrix} 1 & \\ & p \end{pmatrix}$.

The right hand side is the correspondence we used to define the Hecke operator T_p last week. Therefore, we have an algebraic way of considering the Hecke operators via these operations on modular curves, which we will need in order to calculate the action of Hecke operators on the l-adic cohomology groups.

3 Galois Representations attached to modular forms

Last time, we saw the Eichler-Shimura isomorphism,

$$\beta : S_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))} \cong \mathrm{H}^1(Y_{\Gamma_1(N)}(\mathbb{C})^{an}, \mathrm{Sym}^{k-2} R^1\pi_* \underline{\mathbb{C}_{\mathcal{E}}}) =: W_{\mathbb{C}}$$

As discussed above, this gives a way of recognising the coefficients of modular forms inside some cohomology groups. The idea of Deligne was that if we can change this cohomology group to an etale cohomology group, then we should be able to realise the coefficients inside an l-adic representation, and then hope that it satisfies the properties that we want.

We have a $\mathbb{Q}$-form of $W_{\mathbb{Q}}$ given by

$$W_{\mathbb{Q}} := \mathrm{H}^1(Y_\Gamma(\mathbb{C})^{an}, \mathrm{Sym}^{k-2} R^1\pi_* \underline{\mathbb{Q}}).$$

This also has a $\mathbb{T}(N)$-action. The Artin comparison theorem between singular and etale cohomology gives

$$V := W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \mathrm{H}^1_{et}(Y_{\Gamma,\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1_{et}\pi_* \mathbb{Q}_l)$$

Lemma 3.1. The $\mathbb{Q}_l$-vector space V is a finite dimensional l-adic representation of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ unramified outside lN.

Proof. That this representation is finite dimensional follows from the construction. So it suffices to understand the ramification properties now. To do this, pick a prime $p \nmid Nl$. We have a smooth model

$$s : Y_{\Gamma_1(N)} \to \mathrm{Spec}\,(\mathbb{Z}[1/Nl])$$

of $Y_{\Gamma(N)}$ over $\mathbb{Z}\left[\frac{1}{Nl}\right]$, and $\mathrm{Sym}^{k-2}R^1\pi_*\mathbb{Q}_l$ is l-adic with l prime to the characteristic of the base scheme of s, and so by smooth proper base change,

$$\mathcal{F} := R^1 s_* \mathrm{Sym}^{k-2} R^1_{et}\pi_* \mathbb{Q}_l$$

is a l.c.c. sheaf over $\mathrm{Spec}\,(\mathbb{Z}\,[1/Nl])$, and all of its fibres at geometric points are isomorphic. The fibre $\mathcal{F}_{\overline{\mathbb{Q}_p}}$ is the restriction of V to the corresponding decomposition group, and must be isomorphic as Galois representations to $\mathcal{F}_{\overline{\mathbb{F}_p}}$ which is an unramified representation since it factors through $\mathrm{Gal}\left(\overline{\mathbb{F}_p}/\mathbb{F}_p\right)$. □

Therefore we are reduced to studying the Frobenius and Hecke actions on

$$V|_{G_{\mathbb{Q}_p}} = \mathrm{H}^1_{et}(Y_{\Gamma_1(N),\overline{\mathbb{F}_p}}, \mathrm{Sym}^{k-2} R^1_{et}\pi_* \mathbb{Q}_l).$$

Now, we would like to explicitly write out the Hecke operator T_p in terms of a correspondence with $Y_{\Gamma(N;p)}$ over $\mathbb{F}_p$, and therefore relate it to the Frobenius.

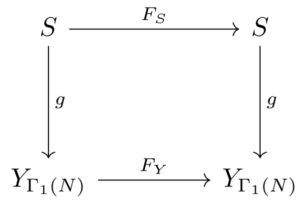
3.1 Frobenius and Étale Cohomology

Quickly, we review the notion of Frobenius acting on étale cohomology, and how this can be related to the Frobenius acting on characteristic p schemes, as well as the connection with moduli spaces.

General principles of étale cohomology say that for a variety $X/\mathbb{F}_p$, the pullback of

$$F : X \longrightarrow X$$
$$x \mapsto x^p : \mathcal{O}_X \longmapsto \mathcal{O}_X$$

on cohomology agrees with the Galois action of the geometric Frobenius Frob_p^{-1}. Let $E \to S$ be any elliptic scheme in characteristic p. By the commutativity of the diagram

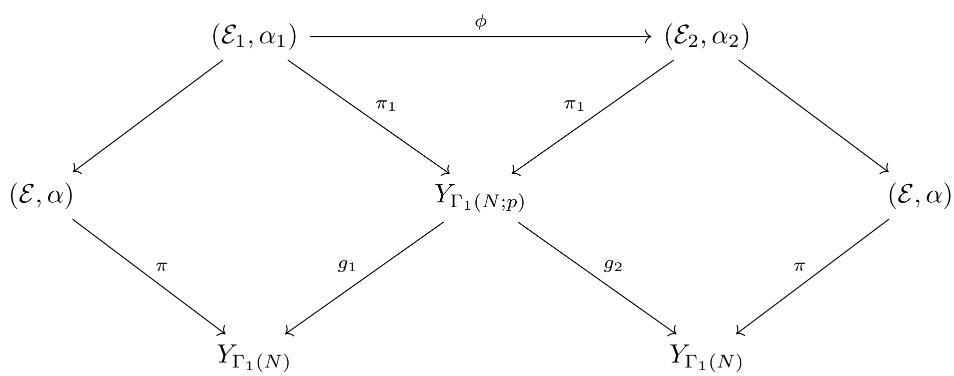


implies that $g^*F_Y^*\mathcal{E} = F_S^*g^*\mathcal{E} = (g^*\mathcal{E})^{(p)}$, and so the morphism F_Y induces the natural transformation

$$\begin{aligned} F : F_N &\longrightarrow F_N \\ (E, \alpha) &\longmapsto \left(E^{(p)}, \alpha^{(p)}\right). \end{aligned}$$

3.2 Action of the Hecke Operator

We are interested in the action of T_p on the cohomology group $\mathrm{H}^1(Y_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1\pi_*\mathbb{Q}_l)$. We need to be careful with twisting the sheaf though so it it not just $g_{1,*}g_2^*$.



The action is actually $T_p = g_{1,*}\phi^* g_2^*$

3.3 The Eichler-Shimura Relation

Theorem 3.2. $\phi_p = \mathrm{Frob}_p^{-1}$ be the Galois geometric Frobenius acting on V. Then

$$T_p = \phi_p + \langle p \rangle p^{k-1} \phi_p^{-1} \in \mathrm{End}_{\mathbb{Q}_l}(V).$$

Proof. We have shown that $\phi_p = F^* \in \mathrm{End}_{\mathbb{Q}_l}(V)$. The idea is to split $Y_{\Gamma_1(N;p),\mathbb{F}_p}$ into two components corresponding to whether the subgroup scheme defined by α is connected or étale, and then compute on each component separately.

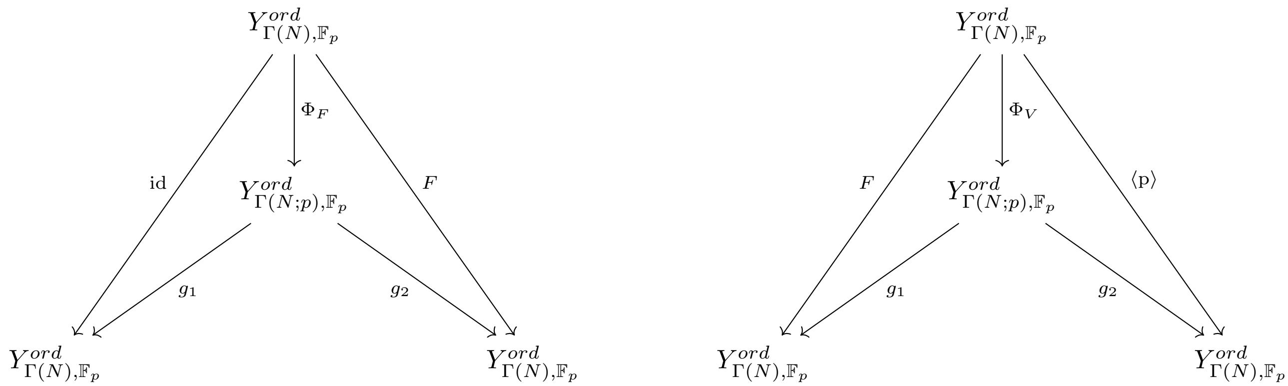
For every pair $(E, \alpha) \in F_N(S)$ over $S/\mathbb{F}_p$, we have a distinguished diagram $(E, E^{(p)}, \alpha, \alpha^{(p)}, F)$, given by the p-isogeny defined by the Frobenius $F : E \to E^{(p)}$. We call the dual isogeny the Verschiebung $V : E^{(p)} \to E$, which fits into the diagram $(E^{(p)}, E, \alpha^{(p)}, p\alpha, V)$.

Therefore we get two natural transformations $\Phi_F, \Phi_V : F_N|_{\underline{\mathrm{Sch}}_{\mathbb{F}_p}} \to F_{N,p}|_{\underline{\mathrm{Sch}}_{\mathbb{F}_p}}$, and two corresponding maps

$$\begin{aligned} \Phi_F, \Phi_V :& Y_{\Gamma(N),\mathbb{F}_p} \to Y_{\Gamma(N;p),\mathbb{F}_p}. \\ \Phi_F :& (E, \alpha) \mapsto (E, E^{(p)}, \alpha, \alpha^{(p)}, F) \\ \Phi_V :& (E, \alpha) \mapsto (E^{(p)}, E, \alpha^{(p)}, p\alpha, V) \end{aligned}$$

Consider a diagram $\mathcal{D}$ over $S/\mathbb{F}_p$ such that E/S is an ordinary elliptic scheme (i.e. it's p-torsion subgroup scheme has degree p étale part and degree p connected part). The kernel of the p-isogeny ϕ in $\mathcal{D}$ is either connected or étale. If it is connected, it must be the kernel of Frobenius, so $\mathcal{D} \cong \Phi_F(E, \alpha)$. If it is étale, the dual isogeny must have connected kernel, so ϕ is isomorphic to the Verschiebung, i.e. $\mathcal{D}$ is in the image of Φ_V.

Therefore the images of Φ_F, Φ_V when restricted to the ordinary locus $Y^{ord}_{\Gamma(N),\mathbb{F}_p}$ are disjoint, and we can consider the correspondence for T_p on each component separately. That is, we consider the two correspondences



For the first correspondence, we are pulling back F, which induces ϕ_p on cohomology, and in the second, we are doing the pushforward F_* and the automorphism $\langle p \rangle^*$. Clearly, the composition of F^* and F_* induces multiplication by p since F is a degree p morphism, and so it induces multiplication by p^{k-1} on the cohomology group, so $F_* = p^{k-1}F^*$. Therefore,

$$T_p = \phi_p + \langle p \rangle p^{k-1} \phi_p^{-1}$$

on a dense subset (the ordinary locus), and therefore this holds everywhere. □

For each normalised eigenform $f \in S_k(\Gamma_1(N), \epsilon)$, we have a ring homomorphism

$$\begin{aligned} \mathbb{T} := \mathbb{T}_k(\Gamma_1(N)) &\longrightarrow K_f \\ T_p &\longmapsto a_p(f) \end{aligned}$$

giving the corresponding system of Hecke eigenvalues. For a place $\lambda | l$ of K_f define

$$V_{f,\lambda} := \mathrm{H}^1(Y_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1 \pi_* \mathbb{Q}_l) \otimes_{\mathbb{T}} K_{f,\lambda}.$$

By the Eichler-Shimura isomorphism, and the Artin comparison theorem,

$$\dim_{K_{f,\lambda}} V_{f,\lambda} = \dim_{K_f \otimes_{\mathbb{Q}} \mathbb{C}} \left(\left(S_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))} \right) \otimes_{\mathbb{T}} K_f \right) = 2$$

since each Hecke eigensystem appears with multiplicity 1 in $S_k(\Gamma_1(N))$. Furthermore, by the Eichler-Shimura relation,

$$\det (1 - \phi_p X | V_{f,\lambda}) = 1 - a_p(f) X + \epsilon(p) p^{k-1} X^2.$$

4 Galois representations valued in Hecke rings

4.1 Hecke Rings and Eisenstein Ideals

Fix an isomorphism $\overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. The Hecke ring $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_l$ is a semi local ring with Krull dimension 1. The minimal primes $\mathfrak{p}$ are in one to one correspondence with Galois conjugates of newforms $f_{\mathfrak{p}} \in S_k(M)$ for some $M | N$, each of which has a corresponding Galois representation into some finite extension of $\mathbb{Q}_l$. The maximal ideals $\mathfrak{m}$ are in one to one correspondence with the residual representations, which we will denote

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\overline{\mathbb{F}_l}).$$

In other words, they determine mod l systems of Hecke eigenvalues.

Definition 4.1. We say that a maximal ideal $\mathfrak{m} \subset \mathbb{T} \otimes \mathbb{Z}_l$ is Eisenstein if $\overline{\rho}_{\mathfrak{m}}$ is reducible. Otherwise we say it is non-Eisenstein.

4.2 Combining the Individual Representations

For each minimal prime $\mathfrak{p} \subset \mathfrak{m} \subset \widehat{\mathbb{T}_{\mathfrak{m}}}$, we have a corresponding representation. We would like to glue these together to get a representation valued in $\widehat{\mathbb{T}_{\mathfrak{m}}}$. For this we use the following lemma of Carayol.

Lemma 4.2 ([Gee09], Lemma 3.7)**.** Suppose that $\overline{\rho}$ is absolutely irreducible. Let R be an object of $\mathcal{C}_{\mathcal{O}}$, and $\rho : G \to \mathrm{GL}_n(R)$ be a lifting of $\overline{\rho}$.

1. If $a \in \mathrm{GL}_n(R)$ and $a\rho a^{-1} = \rho$, then $a \in R^{\times}$.
2. If $\rho' : G \to \mathrm{GL}_n(R)$ is another continuous lifting of $\overline{\rho}$ and $\mathrm{tr}\rho = \mathrm{tr}\rho'$, then there is some $a \in \ker\left(\mathrm{GL}_n(R) \to \mathrm{GL}_n(\mathbb{F})\right)$ such that $\rho' = a\rho a^{-1}$.
3. If $S \subset R$ is a closed subring with $S \in \mathrm{ob}\mathcal{C}_{\mathcal{O}}$ and $\mathfrak{m}_S = \mathfrak{m}_R \cap S$, and if $\mathrm{tr}\rho(G) \subset S$, then there is some $a \in \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(\mathbb{F}))$ such that $a\rho a^{-1} : G \to \mathrm{GL}_n(S)$.

Proposition 4.3. Let $\mathfrak{m} \subset \mathbb{T}_k$ be a non-Eisenstein maximal ideal. Then there is a representation

$$\rho_{\mathfrak{m}}^{mod} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\widehat{\mathbb{T}_{k,\mathfrak{m}}})$$

which is unramified $p \nmid Nl$ and furthermore

$$\mathrm{tr}\rho_{\mathfrak{m}}^{mod}(\mathrm{Frob}_p) = T_p, \det \rho_{\mathfrak{m}}^{mod}(\mathrm{Frob}_p) = p^{k-1}.$$

Thus we get a map $R_{\overline{\rho}_{\mathfrak{m}}} \to \widehat{\mathbb{T}_{k,\mathfrak{m}}}$.

Proof. The minimal primes $\mathfrak{p} \subset \mathfrak{m} \subset \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_l$ correspond to maps

$$\widehat{\mathbb{T}_{k,\mathfrak{m}}} \to \overline{\mathbb{Q}_l},$$

and gives a corresponding Galois representation valued in $\overline{\mathbb{Q}_l}$. We also have that

$$\widehat{\mathbb{T}_{k,\mathfrak{m}}} \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}_l} \xrightarrow{\sim} \prod_{\mathfrak{p}} \overline{\mathbb{Q}_l}.$$

If we take the sum of the attached Galois representations, we get a representation

$$\prod \rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\widehat{\mathbb{T}_{k,\mathfrak{m}}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}_l}) = \prod_f \mathrm{GL}_2(\overline{\mathbb{Q}_l}).$$

These can be conjugated into $\prod_f \mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}_l}})$, so that each one has residual representation equal to (not just conjugate to) $\overline{\rho}_{\mathfrak{m}}$. More precisely, we have two equivalent representations

$$\prod_f \overline{\rho_f} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \prod_f \mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}_l}}) \to \prod_f \mathrm{GL}_2(\overline{\mathbb{F}_l})$$

$$\prod_f \overline{\rho_{\mathfrak{m}}} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m}) \to \prod_f \mathrm{GL}_2(\overline{\mathbb{F}_l}).$$

Therefore we can individually conjugate each ρ_f such that these two representations are each, and so $\prod \rho_f$ has image in $\mathrm{GL}_2(S)$ where

$$S = \left\{ x \in \widehat{\mathbb{T}_{\mathfrak{m}}} \otimes_{\mathbb{Z}_l} | \overline{x} \in \mathbb{T}/\mathfrak{m} \right\}.$$

Now $\widehat{\mathbb{T}_{\mathfrak{m}}} \subset S$ satisfy the conditions for part (3) of Lemma 4.2, so we can conjugate to get a representation valued in $\widehat{\mathbb{T}_{k,\mathfrak{m}}}$. ∎

5 Modular Forms over Totally Real Fields

Let F be a totally real field. Then we can do the same construction as above for Hilbert modular forms attached to this totally real field.

Proposition 5.1 ([Gee09], page 28)**.** Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ of weight (k,η). Then there is a CM field L_π which contains the eigenvalues of T_ν and S_ν on $\pi_\nu^{\mathrm{GL}_2(\mathcal{O}_{F_\nu})}$ for each finite place ν at which π_ν is unramified. Furthermore, for each finite place λ of L_π there is a continuous irreducible Galois representation

$$r_\lambda(\pi) : G_F \to \mathrm{GL}_2(\overline{L}_{\pi,\lambda})$$

such that

1. if π_ν is unramified and $\nu \nmid \mathrm{char}(\lambda)$, then $r_\lambda(\pi)|_{G_{F_\nu}}$ is unramified, and the characteristic polynomial of Frob_ν is
$$X^2 - t_\nu X + (\#k(\nu))\, s_\nu.$$
2. For all finite places ν not dividing the residue characteristic of λ,
$$\mathrm{WD}\left(r_\lambda(\pi)|_{G_{F_\nu}}\right)^{F-ss} \cong \mathrm{rec}_{F_\nu}\left(\pi_\nu \otimes |\det|^{-1/2}\right).$$
3. If ν divides the residue characteristic of λ then $r_\lambda(\pi)|_{G_{F_\nu}}$ is de Rham with τ HT weights $\eta_\tau, \eta_\tau + k_\tau - 1$, where $\tau : F \hookrightarrow \overline{L}_\pi \subset \mathbb{C}$ is an embedding lying over ν. Furthermore, if π_ν is unramified, $r_\lambda(\pi)|_{G_{F_\nu}}$ is crystalline.
4. The representation is odd.

5.1 Local-Global Compatibility for Modular Forms

Suppose that $f \in S_k(\Gamma_1(N), \epsilon)$. Then [LW14],

1. If $p \nmid N$, $\pi_{f,p}$ is unramified principal series with Satake parameters equal to the roots of $X^2 - a_pX + \epsilon(p)p^{k-1}$.
2. If $p^r||N$, and $p^r||\mathrm{cond}(\epsilon)$, $\pi_{f,p}$ is irreducible principal series $\pi(\chi_1,\chi_2)$ where χ_1 is unramified and $\chi_1(p) = a_p(f)/p^{(k-1)/2}$, and $\chi_2 = \epsilon_p/\chi_1$, where ϵ_p is the p-part of the nebentypus, thought of as a character on $\mathbb{Q}_p^\times$.
3. If $p||N$ and $p \nmid \mathrm{cond}(\epsilon)$, then $\pi_{f,p}$ is $\mathrm{St} \otimes \chi$, an unramified twist of the Steinberg representations, with $\chi(p) = a_p(f)/p^{(k-2)/2}$.
4. Otherwise, $\pi_{f,p}$ is supercuspidal.

The first two possibilities correspond to completely reducible Weil-Deligne representations, the third to decomposable but irreducible WD reps, and the final possibility gives indecomposable WD reps. We know the action of Frobenius, so let's determine what this means for the action of inertia. Using the fact that

$$\mathrm{rec}\left(\pi(\chi_1,\chi_2)\right) = (\chi_1 \oplus \chi_2, 0)$$
$$\mathrm{rec}\left(\mathrm{St}\right) = \left(|\cdot|^{1/2} \oplus |\cdot|^{-1/2}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

we get that the cases above give rise to representations $\rho_{f,l}|_{I_{\mathbb{Q}_p}}$,

1. trivial representation;
2. $\begin{pmatrix} 1 & 0 \\ 0 & \epsilon_p \end{pmatrix}$;

3. $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
4. something indecomposable.

6 Algebraic Geometry Background

Definition 6.1.
- The category of locally constant constructible $\mathbb{Z}_l$-sheaves over a scheme S is the category of projective systems of sheaves $(\mathcal{F}_n)_{n\in\mathbb{N}}$ on the etale site of S_{et} each of which is a locally constant sheaf of $\mathbb{Z}/(l^n)$-modules of finite type such that for $n \leq m$, $\mathcal{F}_m \otimes \mathbb{Z}/(l^n) \xrightarrow{\sim} \mathcal{F}_n$.
- The category of l.c.c. $\mathbb{Q}_l$-sheaves is the quotient of this category by the subcategory of $\mathbb{Z}_l$-sheaves annihilated by some power of l.

Theorem 6.2 (Smooth Proper Base Change)**.** Let $\pi : Y \to X$ be proper and smooth, and $\mathcal{F}$ be a l.c.c. sheaf on Y_{et} with torsion prime to char(X). Then $\forall i \geq 0$, $R^i\pi_*\mathcal{F}$ is l.c.c., and in particular, if X is connected,

$$\mathrm{H}^i(Y_{\overline{x}}, \mathcal{F}|_{Y_{\overline{x}}}) = (R^i\pi_*\mathcal{F})_{\overline{x}}$$

are isomorphic for all geometric points of X.

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