Langlands Seminar

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1 Talk 1 - Introduction and Organisation

This talk is intended to be a taster/organisational talk, to show what there is to learn, how things are related, and what the Langlands Program is. Words in red are potential topics that we may wish to explore further.

1.1 Galois Representations

As a number theorist, an object of central importance is that of Galois representations. I could say that most fundamental is integer solutions to polynomial equations, or motives, however these have very quickly given rise to the study of Galois groups and consequently Galois representations.

Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$, we may wish to study Artin representations:

$$G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$$

These have finite image by a topological argument. Another possibility is to study l-adic representations. These arise from étale cohomology. Inside the absolute Galois group we can find copies of the absolute Galois groups of $\mathbb{Q}_p$ for all primes p, and thus by restriction we get representations of these local Galois groups. When $l \neq p$ the l-adic representations of $G_{\mathbb{Q}_p}$ are well-understood, and when $l = p$ this is the realm of p-adic Hodge theory.

In both the l-adic and p-adic cases, it is actually possible to construct a Weil-Deligne representation, which is a more combinatorial object which is independent of topology, and in fact there is an equivalence of categories between l-adic representations of $G_{\mathbb{Q}_p}$ and a certain easy to describe sub-category of Weil-Deligne representations (the l-integral ones, where the eigenvalues of Frobenius all have absolute value 1).

We now consider global Galois representations. Given a global Galois representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}(V)$$

where V is a finite dimensional vector space over $\overline{\mathbb{Q}_l}$, we get by restriction a family of local Galois representations. Suppose that R arises as a subquotient of the étale cohomology of some smooth projective variety $X/\mathbb{Q}$, then it satisfies some nice properties

1. R is unramified outside a finite set of primes.
2. The restriction of R to $\mathbb{Q}_l$ is de Rham.

There is a

Conjecture 1.1 (Fontaine-Mazur)**.** Suppose that $R : G_{\mathbb{Q}} \to \mathrm{GL}(V)$ is an irreducible l-adic representation satisfying the above two properties. Then it comes from the étale cohomology of a smooth projective variety.

1.2 L-functions

One convenient way of packaging Galois representations is via the theory of (motivic) L-functions. We can associate an L-factor to a Weil-Deligne representation by setting

$$L((r,N),X) = \det\left(1 - X\mathrm{Frob}_p\right)|_{V^{I_{\mathbb{Q}_p}, N=0}}.$$

We can also define local Γ- and ϵ-factors. Given a global Galois representation, we can define the L-function of R to be

$$L(R,s) = \prod_p L(WD_p(R), p^{-s}),$$

as well as global Γ- and ϵ-factors. This defines a function convergent on some subset of $\mathbb{C}$. For example, the Riemann zeta function, the Dirichlet L-functions, Artin L-functions, and the L-functions of elliptic curves all arise in this way. Furthermore, by the Cebotarev density theorem, the L-function determines the representation up to semi-simplification.

These L-functions are conjectured to satisfy 2 important properties:

- Meromorphic continuation
- Functional equation

However, these seem extremely difficult to prove. For example, proving this for elliptic curves was only done by Wiles and others in the proof of modularity. One need only look at the Birch and Swinnerton-Dyer conjecture to realise the importance of understanding these properties! For me, the first major utility of the Langlands Program is to give a structured way of approaching the proof of these properties.

1.3 Automorphic Representations

Let's consider the case where $R : G_{\mathbb{Q}} \to \mathrm{GL}_1(\overline{\mathbb{Q}_l})$. Class field theory gives an isomorphism

$$\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0} \to G^{ab}_{\mathbb{Q}}.$$

And therefore the collection of l-adic Galois representations of dimension 1 is in bijection with the collection of l-adic 1-dimensional representations of $\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0}$. Therefore there is a relationship between Galois representations and functions on an adelic group. This relationship is the first example of the Langlands correspondence, and so we need to come up with a general notion of a "nice function on an adelic group" that will hopefully correspond to Galois representations.

The notion that we come up with is that of an automorphic representation. However it is easier to start by understanding automorphic forms. Automorphic forms are essentially nicely behaved functions on the adelic quotient

$$G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})$$

where G is some reductive algebraic group. The space of such functions (satisfying and additional cuspidality condition), often denoted $\mathcal{A}^0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is almost a representation of the adelic group $G(\mathbb{A})$ by right translation, but this doesn't strictly preserve the 'nice' properties of the functions, and so actually we get a representation for each $G(\mathbb{Q}_p)$ with p finite, and we get a $(\mathfrak{g}, K)$-representation at the infinite place (the Langlands classification helps give an understanding of such representations at the infinite place). If we don't make the assumption of cuspidality we also include the Eisenstein series which form a continuous part of the spectrum, but can be built from cuspidal building blocks on smaller groups, which is the philosophy of cusp forms.

An automorphic representation of $G(\mathbb{A})$ is an irreducible $G(\mathbb{A}^\infty)\times(\mathfrak{g}, K)$-module which is a constituent of $\mathcal{A}^0(G(\mathbb{Q}), G(\mathbb{A}))$. We can further break this up according to the infinitesimal character (related to the $(\mathfrak{g}, K)$-representation). One very nice property of automorphic representations (Flath's Theorem) is that they can be written as restricted tensor products

$$\pi = \bigotimes_\nu \pi_\nu$$

where each π_p is an irreducible smooth representation of $G(\mathbb{Q}_p)$, and π_∞ is an irreducible admissible $(\mathfrak{g}, K)$-module. This means that a first step towards understanding automorphic representations is to understand these local components. Furthermore, almost all of the local components are unramified and so can be understood via the Satake isomorphism. The more general theory of smooth representations of p-adic groups is the subject of the Bernstein-Zelevinsky classification

How is this related to the classical theory of modular forms? When $G = \mathrm{GL}_2$, and the infinitesimal character is of a particular form, each cuspidal automorphic representation has a canonical line inside it which is fixed under the action of some open compact subgroup U, so we get a fixed function on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Now, the theory of strong approximation there is an isomorphism

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/U = \bigsqcup_i \Gamma_i\backslash\mathrm{GL}_2(\mathbb{R})$$

and via this isomorphism the particular function we chose is sent to (something very related to) a holomorphic cusp form.

How about other classical automorphic forms? Siegel modular forms correspond to automorphic representations of $\mathrm{GSp}_{2n}(\mathbb{A})$ in much the same way, and Hilbert modular forms correspond to automorphic representations of $\mathrm{GL}_{2/F}$ where F is now some totally real field.

To an automorphic representation (and some additional data), we can construct an automorphic L-function, as well as Γ- and ϵ-factors. For these L-functions it is possible (although still difficult - via the Langlands-Shahidi method) to prove meromorphic continuation and functional equation as are expected for motivic L-functions. These have been constructed by a few different methods, most notably the Rankin-Selberg method.

1.4 Langlands' Conjectures

The previous section suggests a possible way of proving the properties that we want for motivic L-functions. First of all we look at the local situation: there the conjecture is (roughly) that there should be a bijection (called the local reciprocity map) between

$$\{\text{irred. admiss. reps of } G(\mathbb{Q}_p)\} \leftrightarrow \left\{\text{Frob. ss WD-reps } W_{\mathbb{Q}_p} \to {}^LG(\mathbb{C})\right\}$$

The group ${}^LG$ on the right is the Langlands dual group, which is constructed from G, and in some simple cases (e.g. for $G = \mathrm{GL}_n$) is actually just equal to G, and the elements of the set on the right are called L-parameters. This bijection should carry over all of the similar constructions, i.e. the conductor, L-function, ϵ-factor and so on.

For the global conjecture, we need very hard conjectures to even state the conjectured reciprocity. We assume the existence of a group $\mathcal{L}_F$ called the Langlands group which is related to the theory of Tannakian categories. This is supposed to replace the role of the Weil-Deligne group in the local theory. Then there is (again very roughly) a bijection between automorphic representations of G and global L-parameters:

$$\{\text{irred. autom. reps of } G(\mathbb{A})\} \leftrightarrow \left\{\mathcal{L}_F \to {}^LG(\mathbb{C})\right\}$$

In practice there are more conditions to put onto either side, and we still do not get a bijection, but rather a surjection with finite fibres called L-packets, which are generally quite well understood. The objects on the right are very related to Galois representations, and in fact relationships of this form are expected to preserve L-functions and so on, and have been used in some cases to prove all the nice properties of motivic L-functions that we want.

Neither sides of these bijections are well-understood - it is just as interesting a problem to understand the structure of the theory of automorphic representations as it is to understand the Galois side. The statement also suggests another fundamental part of Langlands' conjectures: since the group $\mathcal{L}_F$ is independent of G, we can move between L-parameters using homomorphisms ${}^LG \to {}^LH$, which should then correspond to a relationship between the automorphic forms. This is the notion of functoriality.

1.5 Functoriality

Here are some examples of the principle of functoriality at work:

- (Base Change) If G is defined over a field F, there is always a natural embedding

$$^{L}G \to {}^{L}\mathrm{Res}_{E/F}G$$

for any finite extension of fields E/F. This then translates to the expectation that for any automorphic representation of $G(\mathbb{A}_F)$, there should be a corresponding automorphic representation of $G(\mathbb{A}_E)$ with related properties. A classical example of this is the Doi-Naganuma lifting from modular forms to Hilbert modular forms.

- (Symmetric Powers) There is a symmetric power homomorphism $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$, which give rise to symmetric power liftings. The existence of functorial lifts for symmetric powers implies the Ramanujan-Peterson conjectures.

- (Jacquet-Langlands) If D is a quaternion algebra over $\mathbb{Q}$, the group $G = D^\times$ is an inner form of GL_2 and so they have the same L-group. This leads to the Jacquet-Langlands correspondence between quaternionic automorphic forms and classical modular forms.

- (Endoscopic Classification) For the classical groups, e.g. $\mathrm{SO}_n, \mathrm{Sp}_{2n}, ...$, there are natural maps of their L-groups into some $\mathrm{GL}_N(\mathbb{C})$, and these give rise to the endoscopic transfers which have been constructed by Arthur.

1.6 Proving Cases of Langlands' Conjectures

One method of proving cases of reciprocity is to realise automorphic representations inside the cohomology of Shimura Varieties. Since this cohomology also carries an Galois action, these are a natural common ground of comparison between the two sides. As an example, there are Galois representations attached to automorphic forms on unitary groups. Usually the direction of attaching Galois representations to automorphic forms is easier than the other way round, as can be done by Shimura's general theory.

For proving functoriality, there are a few approaches:

- The Trace Formula: This relates automorphic information in the form of traces of automorphic representations with geometric information in the form of orbital integrals. The most common use of this is rather than explicitly computing either side, one can match up orbital integrals on two different groups that are somehow related (e.g. inner forms). This then gives a matching between the automorphic data. This for example is how the endoscopic classification, Jacquet-Langlands, and solvable base change were resolved.

- Theta Correspondence: This is a correspondence (also called Howe duality) between automorphic representations on two different groups which can be embedded into a single group is a sufficiently nice way (Dual reductive pairs). It is not clear to me at least though what exact instance of functoriality this relates to.

1.7 Relative Langlands Program

This seems to be the intended focus of the IHES Summer School. The idea here is that many interesting invariants of automorphic forms can be expressed as period integrals, which are integrals of the automorphic form over the image of a $H \subset G$ in the quotient $\Gamma\backslash G$.

The relative Langlands Program considers more generally G acting on schemes, and then the is a certain notion of an attached period and so on. This is related to the notion of distinction.

1.8 Leftovers

Things I have not discussed:

1. p-adic deformation on both sides. There are theories of p-adic deformation either through eigenvarieties or Mazur's deformation spaces. The story of how these relate seems very interesting.

2. Function fields: all of these hold for function fields as well, and in this case many of the conjectures are proven!

3. Relations to physics, and the geometric Langlands program