# Fargues-Fontaine Seminar - The Curve

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Heavily based on Section 5 of Chapter 2 in [KDK].

## 1 Notation, Set-up, and Previous Results

Let $E$ be a local field with uniformising element $\pi$ and finite residue field $\mathbb{F}_q$. Let $F/\mathbb{F}_q$ be a perfect valued complete extension for a non-trivial valuation $\nu : F \to \mathbb{R} \cup \{\infty\}$.

Denote by $\mathcal{E}/E$ the unique complete unramified extension of $E$ inducing the extension $F/\mathbb{F}_q$ on residue fields. Then

$$\mathcal{E} = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \middle| x_n \in F \right\}.$$

This has a Frobenius $\varphi$ acting on it, via the Frobenius acting on the extension $F/\mathbb{F}_q$ in the Teichmuller coefficients. We define a subring

$$B_F^b = \left\{ \sum_{n \gg -\infty} [x_n]\, \pi^n \in \mathcal{E} \middle| \nu(x_n) \text{ is bounded below} \right\}.$$

On this ring, we have valuations $\nu_r, r \geq 0$, given by

$$\nu_r(x) = \inf_{n \in \mathbb{Z}} \{\nu(x_n) + nr\}.$$

We define $B_F$ to be the completion of $B_F^b$ with respect to these valuations for $r > 0$. This inherits an action of $\varphi$.

### 1.1 Ad Hoc Alg Geom Definitions

We make a number of 'geometric' definitions which we want to later make genuine.

**Definition 1.1.** The 'set of closed points of $Y$', $|Y|$, is defined to be the set of primitive irreducible elements modulo multiplication by an element of $W_{\mathcal{O}_E}(\mathcal{O}_F)$. This comes with a well defined degree function coming from the degree of the primitive element.

**Definition 1.2.** The 'set of effective divisors on $Y$' is given by

$$\mathrm{Div}^+(Y) = \left\{ \sum_{\mathfrak{m}} a_{\mathfrak{m}}[\mathfrak{m}] \middle| \forall I \subset (0,1) \text{ compact, } \{\mathfrak{m} | a_{\mathfrak{m}} \neq 0 \text{ and } ||\mathfrak{m}|| \in I\} \text{ if finite} \right\}.$$

We learnt that when? there is an injection

$$\mathrm{div} : B \setminus 0 / B^\times \hookrightarrow \mathrm{Div}^+(Y)$$

given by sending $f$ to $\sum_{\mathfrak{m}\in|Y|}\mathrm{ord}_{\mathfrak{m}}(f)[\mathfrak{m}]$, where $\mathrm{ord}_{\mathfrak{m}}$ refers to the normalised valuation on $\widehat{B_{\mathfrak{m}}}$. Note that here we consider $\mathfrak{m}$ as an ideal of $B$, which is the ideal generated by the corresponding primitive element of $W_{\mathcal{O}_E}(\mathcal{O}_F)$. We also analysed the 'divisors on the quotient $|Y|/\varphi^{\mathbb{Z}}$', given by

$$\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}) = \left\{D \in \mathrm{Div}^+(Y) | \phi^* D = D\right\}.$$

The above injection gives another injection

$$\mathrm{div}: \left(\bigcup_{d\geq 0} P_d \setminus \{0\}\right)/E^\times \to \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$$

which is an isomorphism if $F$ is algebraically closed.

We defined

$$P_{F,d} = B_F^{\varphi=\pi^d}, P_F = \sum_{n\geq 0} P_{F,d}$$

# 2 The Fargues-Fontaine Curve

**Definition 2.1.** The (schematic) Fargues-Fontaine curve over $F$ is defined as

$$X_F := \mathrm{Proj}\left(\bigoplus_{d\geq 0} B_F^{\varphi=\pi^d}\right).$$

We will try and understand this scheme bit by bit.

1. Points
2. Residue Fields
3. Open covers
4. Local rings

Until explicitly stated otherwise, assume $F = \overline{F}$.

The main technical tool that we require in order to prove the Theorem is the Fundamental exact sequence.

**Theorem 2.2.** $F = \overline{F}$. Let $t_1, ..., t_n \in P_1$ correspond to points $\mathfrak{m}_1, ..., \mathfrak{m}_n \in |Y|$ in different $\varphi^{\mathbb{Z}}$-orbits. Let $a_1, ..., a_n \in \mathbb{N}_{\geq 1}$ and set $d = \sum_i a_i$. Then for $r \geq 0$, there is an exact sequence of $E$-vsps

$$0 \to P_r \prod_{i=1}^n t_i^{a_i} \to P_{d+r} \xrightarrow{u} \prod_{i=1}^n B_{dR,\mathfrak{m}_i}^+/\mathfrak{m}_i^{a_i} \to 0.$$

## 2.1 Points

The first thing we'd like to do is calculate the points of the curve. We have a guess, since the points of $|Y_F|/\varphi^{\mathbb{Z}}$ are in bijection with elements of $(P_1 \setminus \{0\})/E^\times$. Also, each of these elements spans a homogeneous prime ideal $Pt \subset P$ and so gives a point in $X_F$. That is, we have a map

$$\begin{aligned}\alpha : (P_1 \setminus \{0\})/E^\times &\to X_F\\ tE^\times &\mapsto \infty_t := (t).\end{aligned}$$

**Proposition 2.3.** The map $\alpha$ is injective and its image is the closed points of $X_F$.

*Proof.* Injectivity is clear, since $(t) \cap P_1 = P_0 t = Et$ so the ideal uniquely determines the $E$-line $P_1$. For the image, first note that since $P$ is graded factorial with irreducibles in degree 1, every prime ideal is either $(0)$ or contains some ideal $(t)$. Therefore it suffices to look at $P/(t)$ and prove this has no non-trivial homogeneous prime ideals.

We now apply the Fundamental Exact Sequence, with just the point $t$, which says that

$$0 \to P_r t \to P_{r+1} \to C_{\mathfrak{m}} \to 0.$$

Therefore

$$P/(t) = E \oplus \bigoplus_{r \geq 1} C_{\mathfrak{m}} = \{ f \in C_{\mathfrak{m}}[T] | f(0) \in E \} .$$

The only homogeneous primes are the zero and irrelevant ideals. □

## 2.2 Local Rings and Residue Fields

Now, let's calculate the residue field and local ring at a point $tE^\times \leftrightarrow (t) \leftrightarrow \mathfrak{m}$. Thinking of $|Y| \subset \operatorname{Spec}(B)$, we denoted the residue field at $\mathfrak{m}$ in $\operatorname{Spec}(B)$ by $C_{\mathfrak{m}}$, and the local ring by $B^+_{dR,\mathfrak{m}}$ so that

$$\theta_{\mathfrak{m}} : B^+_{dR,\mathfrak{m}} := \widehat{B_{\mathfrak{m}}} \twoheadrightarrow C_{\mathfrak{m}}.$$

Again, we will prove that these hold for $X$ as well. At a prime $(t)$, the local ring $\mathcal{O}_{X,\infty_t}$ is given by

$$\mathcal{O}_{X,\infty_t} = P_{(t),0} = \bigcup_{y \in P_d \setminus tP_{d-1}} \frac{1}{y} P_d.$$

Since $y \notin tP_{d-1}$, $\operatorname{ord}_{\mathfrak{m}}(y) = 0$, and so $y \in \left( B^+_{dR,\mathfrak{m}} \right)^\times$, which means that we can consider $\mathcal{O}_{X,\infty_t} \subset B^+_{dR,\mathfrak{m}}$. From the FES, we get

$$0 \to \frac{t}{y} P_{d-1} \to \frac{1}{y} P_d \to C_{\mathfrak{m}} \to 0.$$

And so the embedding $\mathcal{O}_{X,\infty_t} \hookrightarrow B^+_{dR,\mathfrak{m}}$ is an embedding of DVRs sending a uniformising element to a uniformising element and inducing an isomorphism on residue fields, so it is an isomorphism. Note that this also means that the ad hoc definition of the divisors of functions

## 2.3 Open Covers

We would like to construct an affine cover of the scheme so that we might be able to understand it a little better schematically (many definitions in scheme theory go via an open cover). Thinking of $X_F$ like $\mathbb{P}^1$ suggests that a good open cover would come from remove each point to form copies of $\mathbb{A}^1$.

This logic leads us to examine the set $D^+(t) := \{ \mathfrak{p} \not\supseteq (t) : \text{homogeneous prime ideals} \} \subset X_F$. We have seen above that

$$V^+(t) = X_F \setminus D^+(t) = \{ \infty_t \} .$$

By usual algebraic geometry,

$$D^+(t) = \operatorname{Spec} \left( P \left[ \frac{1}{t} \right]_0 \right) = \operatorname{Spec} \left( B \left[ \frac{1}{t} \right]^{\varphi=1} \right)$$

**Lemma 2.4.** Suppose $F = \overline{F}$. For $t \in P_1 \setminus \{0\}$, the ring $P \left[ \frac{1}{t} \right]^{\varphi=1}$ is a PID. However, contrary to the case of $\mathbb{P}^1$, where the corresponding ring is $E \left[ \frac{x_0}{x_1} \right]$, it is not Euclidean with the degree valuation, but only almost Euclidean (the remainder has degree non-strictly less than the degree of the divisor).

*Proof.* This is essentially the same as the proof above but applied to $P\left[\frac{1}{t}\right]_0$. We can factor any element into irreducibles of the form $\frac{t'}{t}$ where $t' \notin Et$. To exhibit the ideal generated by $\frac{t'}{t}$ as a maximal ideal of $B\left[\frac{1}{t}\right]_0$, we clearly want it to be the kernel of the map to the residue field at the prime corresponding to $t'$. So, consider the morphism

$$B\left[\frac{1}{t}\right]_0 \to C_{\mathfrak{m}'}$$

where $\mathfrak{m}'$ is the element of $|Y|$ attached to $t'$, thought of as an ideal of $B$. This morphism exists because $\theta_{\mathfrak{m}'}(t) \neq 0$ by the identification

$$\left(P_1 \setminus \{0\}\right)/E^\times \xrightarrow{\sim} |Y|/\varphi^{\mathbb{Z}}.$$

By the FES, this is surjective with kernel generated by $t'/t$.

For the statement about almost Euclidean, suppose that $x, y \in P\left[\frac{1}{t}\right]_0 \setminus \{0\}$ have degrees $d > d'$ respectively. Then we can write them as

$$x = \frac{\alpha}{t^d}, y = \frac{\beta}{t^{d'}}, \text{ where } \alpha \in P_d \setminus tP_{d-1}, \beta \in P_{d'} \setminus tP_{d'-1}.$$

The images of $\alpha, \beta$ in $P/tP = \{f \in C_{\mathfrak{m}}[T] | f(0) \in E\}$ are given by $\theta_{\mathfrak{m}}(\alpha)T^d, \theta_{\mathfrak{m}}(\beta)T^{d'}$. We want to take element $\gamma \in P_{d-d'}$ such that $\gamma \equiv \frac{\theta_{\mathfrak{m}}(\alpha)}{\theta_{\mathfrak{m}}(\beta)}T^{d-d'} \in P/tP$, but we can only guarantee the existence of this since $d' < d$ (this is where we differ from the usual situation). We now take $\delta \in P_{d-1}$ such that $\alpha = \gamma\beta + t\delta$, so that

$$x = \left(\frac{\gamma}{t^{d-d'}}\right) y + \frac{\delta}{t^{d-1}}.$$

We can now apply the usual recursion to get an almost Euclidean factorisation. □

**Corollary 2.5.** The Fargues-Fontaine curve $X_F$ is an integral Noetherian regular scheme of dimension 1 over $\operatorname{Spec}(E)$. However, it is not of finite type.

*Proof Of Corollary.* A PID is an regular Noetherian ring of Krull dimension 1. Also $P$ is a domain, so $X_F$ is integral. The residue fields are algebraically closed over $E$, so are not finite extensions, so $X_F$ is not of finite type. □

## 3 The Fundamental Exact Sequence

**Theorem 3.1.** $F = \overline{F}$. Let $t_1, ..., t_n \in P_1$ correspond to points $\mathfrak{m}_1, ..., \mathfrak{m}_n \in |Y|$ in different $\varphi^{\mathbb{Z}}$-orbits. Let $a_1, ..., a_n \in \mathbb{N}_{\geq 1}$ and set $d = \sum_i a_i$. Then for $r \geq 0$, there is an exact sequence of $E$-vsps

$$0 \to P_r \prod_{i=1}^n t_i^{a_i} \to P_{d+r} \xrightarrow{u} \prod_{i=1}^n B^+_{dR,\mathfrak{m}_i}/\mathfrak{m}_i^{a_i} \to 0.$$

We now need to prove this.

*Proof.* Injectivity is obvious.

Exactness in the middle is due to the fact that in $B \setminus \{0\}$, $f$ is a multiple of $g$ iff $\operatorname{div}(f) \geq \operatorname{div}(g)$. So it remains to prove surjectivity. By induction, reduce this to the case $n = 1, a_1 = 1$, so we need to prove that

$$\theta_{\mathfrak{m}} : B^{\varphi=\pi} \to C_{\mathfrak{m}}$$

To prove this, we use the description of $B^{\varphi=\pi}$ via Lubin-Tate formal groups. Recall from Arun's talk that there is an isomorphism

$$\begin{aligned} \mathcal{G}(\mathcal{O}_F) = (\mathfrak{m}_F, +_{\mathcal{G}}) &\longrightarrow B^{\varphi=\pi} \\ \epsilon &\longmapsto \log_Q\left([\epsilon]_Q\right) \end{aligned}$$

In Alex's talk, we saw that

$$\varprojlim_{\varphi} \mathcal{G}(\mathcal{O}_{C_{\mathfrak{m}}}) \xrightarrow{\sim} \mathcal{G}(\mathcal{O}_F).$$

Since we are assuming $F$, and therefore also $C_{\mathfrak{m}}$, to be algebraically closed, we get an isomorphism

$$\varprojlim_{x \mapsto x^q} (1 + \mathfrak{m}\mathcal{O}_{C_{\mathfrak{m}}}) \xrightarrow{\sim} \varprojlim_{\varphi} \mathcal{G}(\mathcal{O}_{C_{\mathfrak{m}}}).$$

Working through the identifications, we see that $\theta_{\mathfrak{m}} : \varprojlim(1 + \mathfrak{m}\mathcal{O}_{C_{\mathfrak{m}}}) \to C_{\mathfrak{m}}$ is simply given by $(x^{(n)})_{n \geq 0} \mapsto \log(x^{(0)})$. □

# 4 Non-Algebraically Closed Fields

Now we analyse the curve in general. Let $F$ be an arbitrary (perfect) extension of $\mathbb{F}_q$. How does the curve $X_F$ relate to $X_{\widehat{\overline{F}}}$? We will relate them on affine covers.

The morphism of graded algebras $P_F \to P_{\widehat{\overline{F}}}$ induces a morphism

$$\alpha : X_{\widehat{\overline{F}}} \to X_F$$

**Proposition 4.1.** Let $t \in P_{F,1} \setminus \{0\}$. Then $\alpha^{-1}(tP_F) = \{tP_{\widehat{\overline{F}}}\}$. Now restrict $\alpha$ to a morphism

$$\alpha : \operatorname{Spec}\left(B_{\widehat{\overline{F}}}\left[\frac{1}{t}\right]^{\varphi=1}\right) \to \operatorname{Spec}\left(B_F\left[\frac{1}{t}\right]^{\varphi=1}\right).$$

Call these rings $A_{\widehat{\overline{F}}}$ and $A_F$ respectively.The ring $A_F$ is a Dedekind domain, and the maps

$$I \mapsto A_{\widehat{\overline{F}}} I$$
$$J \mapsto J \cap A_F$$

are inverse bijections between non-zero ideals of $A_F$ and non-zero $G_F$-invariant ideals of $B_{\widehat{\overline{F}}}$.

*Proof.* First of all, we prove that

$$(A_{\widehat{\overline{F}}})^{G_F} = \mathrm{H}^0(G_F, A_{\widehat{\overline{F}}}) = A_F$$
$$\mathrm{H}^1(G_F, A_{\widehat{\overline{F}}}) = 0.$$

We pick an $f \in A_F \setminus \{0\}$, and look at the $f$-adic completion of $A_F$. By the above cohomological results,

$$A_F/f^n A_F = \left(A_{\widehat{\overline{F}}}/f^n A_{\widehat{\overline{F}}}\right)^{G_F}.$$

Suppose that $f = u \prod_{i=1}^r f_i^{a_i}$ in $A_{\widehat{\overline{F}}}$. Then the fundamental exact sequence tells us that

$$A_{\widehat{\overline{F}}}/f^n A_{\widehat{\overline{F}}} = \prod_{i=1}^r B^+_{\widehat{\overline{F}}, dR, \widehat{\mathfrak{m}}_i} / \widehat{\mathfrak{m}}_i^{na_i},$$

where $\widehat{\mathfrak{m}}_i$ is the maximal ideal in $\left|Y_{\widehat{\overline{F}}}\right|$ corresponding to $f_i$. Let $\mathfrak{m}_j$ for $1 \leq j \leq s$ be the intersections $\widehat{\mathfrak{m}}_i \cap A_F$ for each of the $G_F$-orbits. By Galois invariance of $f$, we see that the function $a_i$ is constant on the $G_F$-orbits, so we think of it as a function on the $\mathfrak{m}_j$. Sen Theory combined with the Galois descent of Alex's lecture tells us that

$$\left(A_{\widehat{\overline{F}}}/f A_{\widehat{\overline{F}}}\right)^{G_F} = \prod_{j=1}^s B^+_{F, dR, \mathfrak{m}_j} / \mathfrak{m}_j^{na_j}$$

so

$$A_{F,f} = \prod_{j=1}^{s} B_{F,dR,\mathfrak{m}_j}$$

and functors $I \mapsto I^{G_F}$ and $J \mapsto A_{\widehat{\overline{F}},f} J$ define inverse bijections between $G_F$-invariant ideals of $A_{\widehat{\overline{F}},f}$ and ideals of $A_{F,f}$. It now suffices to show that finish off.

□

**Theorem 4.2.**

- For all $x \in |X_F|$, $\alpha^{-1}(x)$ is a finite set of closed points of $\left|X_{\widehat{\overline{X}}}\right|$.
- for $x \in \left|X_{\widehat{\overline{F}}}\right|$, either $\alpha(x)$ is the generic point of $X_F$, in which case $G_F x$ is infinite, or $\alpha(x)$ is a closed point of $X_F$ and $G_F x$ is finite.
- the induced map

$$\left|X_{\widehat{\overline{F}}}\right|^{G_F - fin} / G_F \xrightarrow{\sim} |X_F|\,.$$

From this, we get the following corollary, descending from the algebraically closed case.

**Corollary 4.3.**

1. $X_F$ is a integral Noetherian regular scheme of dimension 1.
2. For $x \in |X_F|$, set $\deg(x) = \#\alpha^{-1}(x)$. Then for $f \in E(X_F)^{\times}$,

$$\deg(\mathrm{div}(f)) = 0.$$

   Thus $X_F$ is a complete curve.
3. For $\mathfrak{m} \in |Y_F|$, define

$$\mathfrak{p}_{\mathfrak{m}} = \left\{ \sum_{d \geq \deg \mathfrak{m}} x_d \in P_F \,\middle|\, x_d \in P_{F,d}, \mathrm{div}(x_d) \geq \sum_{n \in \mathbb{Z}} [\varphi^n(\mathfrak{m})] \right\},$$

   which is a prime homogeneous ideal. Furthermore

$$\begin{aligned} |Y_F|/\varphi^{\mathbb{Z}} &\longrightarrow |X_F| \\ \varphi^{\mathbb{Z}}(\mathfrak{m}) &\longmapsto \mathfrak{p}_{\mathfrak{m}} \end{aligned}$$

   is an isomorphism, and there is an identification $\mathcal{O}_{X_F,\mathfrak{p}_{\mathfrak{m}}} = B^+_{F,dR,\mathfrak{m}}$.

*Proof of Theorem.* □

$$\mathrm{Spec}\,(\mathbb{Z}) \cong \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$$