Equidistribution Problems over Totally Real Fields

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Contents

To all those who struggle with mathematics, you are not alone.

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Abstract

This thesis concerns equidistribution problems over totally real fields. Our main result is extending the work of Khayutin in [Kha17] to prove the joint equidistribution of CM points on quaternion algebras over totally real fields. To aid further results in this area, we give a general treatment of the ergodic theory required to prove equidistribution results of CM points, and some results towards the case of unitary groups, as well as the Kuga-Sato setting.

Chapter 1

Introduction

Given a measure space (X, μ) and a sequence of finite subsets $S_n \subset X$, we say that the sequence is equidistributed if the normalised discrete measures on S_n tend in the weak-* limit to μ. The aim of this thesis is prove equidistribution in a variety of number theoretic settings over a general totally real base field. This appears to be the first systematic use of ergodic theoretic ideas over a base field different than $\mathbb{Q}$, however such extensions are now proving vital in the ongoing work of Andrew Wiles on modularity. In particular, the original parts of this thesis will be the following (see Section 1.2 for a more thorough overview and discussion of how the chapters of this thesis relate and go into the final proofs):

1. Explicate the general ergodic methods of proving equidistribution and joint equidistribution for a wide class of groups over totally real fields. This method has been used many times, however the measure theory is always considered in the case particular to that result - here we treat the general case, and discuss the scope and limitations of the method. Our hope is that such a general treatment should encourage and allow number theorists to approach equidistribution problems (in which they should be very interested) without having to each delve into the detailed ergodic theory and homogeneous dynamics. The main ergodic result we prove and use here is Theorem 3.6.1.

2. Prove a version of [Kha17] over totally real fields, which proves the joint equidistribution of torus orbits on quaternion algebras over $\mathbb{Q}$. This result in particular is useful for the work of Wiles, and so it is this that we give the most attention. See Theorem 2.6.5, which is restated and proven as Theorem 8.2.2.

3. Discussion of possibilities for proving further cases of equidistribution. In particular, we develop much of the background needed in the case of unitary groups,

which is also important for number theoretic applications. There are, however, some fundamental obstacles in this case which we hope to illuminate, in order that people may be motivated to work on these obstructions.

4. Informal discussion of some ideas that we think may be important in future work on equidistribution but have not had time or space to implement. This includes a more automorphic perspective on the ergodic method, which may be useful motivationally for future results.

1.1 Methods of Proof of Equidistribution

There are two parallel schools in proving arithmetic equidistribution problems. The first (see for example [Lin68; CU05; Ein+06; MV06; EMV10]) began with Linnik and his ergodic method of proving equidistribution of integer points on spheres and a number of related problems. The second begins with Weyl's equidistribution criterion, and perhaps is typified by Duke's proof of (a stronger version of) Linnik's Theorem via the theory of automorphic forms (see [Duk88; DS90; Zha05; BB20]).

While the second method became the fashionable method of proving equidistribution, and proves stronger results, the input required is often very difficult to establish (for example Waldspurger's formula [Wal85]), and the method has some serious analytical barriers in further generality (often GRH is assumed in modern applications). On the other hand, while the results of an ergodic argument are often slightly weaker (they are ineffective and require certain splitting conditions), they can often get off the ground with less input (e.g. subconvexity or Brauer-Siegel). To demonstrate this, we consider the problem of equidistribution of points on spheres.

In Duke's approach, the collection of integer points are weighted against arbitrary functions in $L^2(S^2)$ to produce Weyl sums. These are then recognised (by work originating in Maass, see [DS90]) as the Fourier coefficients of modular forms, specifically theta series, which can in turn be estimated by work of Iwaniec extended by Duke.

In Linnik's approach, he constructs an action of a class group on the set of integer points and considers the dynamics of this action. Essentially the key input is his Basic Lemma (see [EMV10, Prop. 2.11] and [WY22]), in which he proves that trajectories under this action do not correlate with each other, and stay relatively separated. To prove this, he translates the problem into one of counting quadratic forms that embed into $\mathbb{Z}^3$, and tackles this with Siegel's Formula for embedding numbers of quadratic forms (this form of Linnik's argument was first explicated in [EMV10]). Interestingly, this input is essentially automorphic, and concerns the Fourier coefficients of genus 2

Siegel modular forms. In a loose sense, therefore, the ergodic method can be seen as a way of producing bounds on many Fourier coefficients for genus 1 (i.e. ordinary) modular forms from easier bounds on a smaller collection of genus 2 Siegel modular forms. Reinterpreted, the method of Linnik is now often referred to as the high entropy method, which uses similar ideas to prove that the correlation between two torus orbits on homogeneous spaces are reasonably small, thus giving traction to general measure theoretic results (see [EL08]).

More recently, these ideas have been applied more generally, but one particular difficulty recurs in many cases. These problems are rephrased as equidistribution on an algebraic group of some kind, and intermediate groups between the torus and the whole group can be a barrier to equidistribution - essentially the points of interest can accumulate around Hecke subvarieties. In a pair of papers [Kha17; Kha19b], Khayutin has tackled two cases of joint equidistribution (where the pairs of an integer point and a fixed translate of it are shown to equidistribute over the product) via a new method of bounding the correlation between torus orbits and Hecke orbits, so as to remove the obstacle of intermediate subgroups. A crucial point which allows the success of such results is the following: joinings of measures are extremely well understood, and completely algebraic. This differs from the single equidistribution case where we may have extremely non-algebraic measures on subvarieties - these non-algebraic measures are really the main obstacle to extending these methods.

The correlations to be bounded in the works of Khayutin are defined as certain integrals over adelic groups. In the case of joint equidistribution on quaternion algebras, these integrals are bounded in an intricate argument via sieve theory, and in the case of Kuga-Sato varieties they are related to Hecke L-functions and bounded using subconvexity. In both cases, the translation from correlations to quantities amenable to analysis by sieve theory or subconvexity are reasonably ad hoc using explicit coordinates and invariant maps (see Sections 2.4 and 6.1.4) available in each situation. We can summarise this method as follows:

$$\begin{array}{ccc} \text{Correlations} & \xrightarrow{\text{Coordinates}} & \text{Explicit Sums} \\ & & \Big\downarrow{\scriptstyle \text{Sieves/Subconvexity}} \\ & & \text{Bounds} \end{array}$$

Our method follows that of Khayutin, which we view abstractly as follows. Given G acting on another linear algebraic group H, we form $G \ltimes H$ which has two commuting left G-actions on it: the first is the diagonal $g \cdot (l,h) = (gl, gh)$ and the second is $g * (l,h) = (lg^{-1}, h)$.[1]

Now, the correlations between torus orbits and the diagonal can be rephrased as a counting problem on

$$G \setminus (G \ltimes H)/T = T \setminus H$$

i.e. the orbit space of T acting on H. In fact, we need to understand the action of the integral elements of the torus T on H. To do this, we lift the problem to the Lie algebra. By decomposing $\mathrm{Lie}(H)$ into representations of the torus T, we get good representatives for the orbits $T \setminus H$. For example in the Kuga-Sato setting this is simply the action of the norm 1 elements of a CM field E on the vector space E.

Counting orbits on the Lie algebra leads to a sum of multiplicative functions on bounded sets of integral elements of CM fields, which are then approached by analytic means (sieve theory in the joint case, Hecke L-functions in the Kuga-Sato case). Our eventual hope, however, is that these sums may be recast in an automorphic way themselves. Since we have an action of a torus on a linear space and are integrating over this action against a test function, there is a hope this may arise from some kind of theta correspondence (and in fact in the Kuga-Sato setting this does occur). Diagrammatically, our aim is to reach the necessary bounds via the lower route:

$$\begin{array}{ccc}
\text{Correlations} & \xrightarrow{\text{Coordinates}} & \text{Explicit Sums} \\
{\scriptstyle \text{Theta Functions/ Spectral Expansion}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Sieves/Subconvexity}} \\
\text{Automorphic Periods} & \xrightarrow{\text{Analytic Theory}} & \text{Bounds}
\end{array}$$

Note that it may be that the bounds required from the automorphic periods could well be proven by methods of sieve theory or subconvexity, however the applicability of these to problems in automorphic forms is a very well-understood field and it would be of significant advantage to know an automorphic replacement for the explicit sums. This is, however, just philosophy, and we now proceed with the concrete task at hand.

1.2 Outline of Thesis

Here, we will outline the sections of this thesis.

Section 1: This outlines the setting of the thesis, historical approaches, and our results.

[1] To see the joint CM case in this framework, consider $G = G^{\Delta} < G \times G$ as the diagonal subgroup and $H = G \times \{e\} < G \times G$. Then the joint CM problem comes from the above problem with the action of G on H by conjugation.

Section 2: This gives the formal statement of the problems that we consider, defines the notation, and states the various collections of assumptions we will assume. It also sets out the specific cases (Kuga-Sato varieties and joint equidistribution over quaternion algebras, unitary groups) to which the general theory will be applied later in the work. Furthermore, it will discuss algebraic tori, and prove necessary results on their embeddings into algebraic groups.

Section 3: Here, we will cover the ergodic background and results needed. The main aim of the section is to reduce questions of equidistribution to an explicit bound on correlations (as well as discuss the issue of intermediate subgroups). For the most part, this background is contained amongst other references in specific cases, however we aim here for generality and applicability, and highlight some adjustments necessary for our results.

Section 4: This short section will prove required volume formulae for toral and Hecke homogeneous sets.

Section 5: This section informally discusses a different approach to mixing that appeared in the original papers on the subject ([EMV10]). It is not necessary for our final results.

Section 6: This section will develop the geometric expansion of the correlation. In the case of mixing, we then reduce this expansion to a form more amenable to analytic methods. This follows the method of [Kha17] closely, however some essential adjustments are required in our more general set-up.

Section 7: In this section, we prove the necessary analytic results to apply to the case of joint equidistribution for totally real fields, which includes a version of the Van der Corput result on points in ellipses and sieve theory developed over the adeles. Both of these results appear to be new.

Section 8: Finally, we will bring together the previous sections to deduce new cases of equidistribution.

Appendices: These are simply adjustments/summaries of the appendices of [Kha17], to which no essential change is required.

1.3 Notation

1. For a number field L, let Σ_L denote the set of places of L. For a fixed set of primes S of $\mathbb{Q}$, let $\Sigma_{L,S}$ denote the primes of L which lie above a prime in S. Unless otherwise stated, F denotes a totally real number field, and K, E denote quadratic CM (i.e. totally imaginary) extensions of F, and M denotes

an arbitrary CM extension of F, and $d = [F : \mathbb{Q}]$. We use the notation ${}^{\sigma}(\cdot)$ for the action of $\sigma \in \mathrm{Gal}\left(\overline{F}/F\right)$. For a set of places S, we often write $S\infty := S \cup \infty$ which is the set consisting of all places in S and all Archimedean places.

2. For any set or product, A, indexed over primes of L, denote by A_S^T the fibre/product (possibly restricted product if S is infinite) over the places of $S \setminus T$, and A_f refers to the restricted product of finite places. For example $(F^{\infty})^{\times} = \mathbb{A}_{F,f}^{\times}$ is the finite ideles, and $F_{\infty} = F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\nu|\infty} F_{\nu}$ is the Archimedean adeles.

3. Let G be an simple algebraic group defined over a number field F and let $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}} G$ which is a $\mathbb{Q}$-simple algebraic group. We define G^{spl} to be the split form of G, and will often abuse notation by using G^{spl} to denote this unique isomorphism class over fields other than F, for example over $\mathbb{Q}$ or $\mathbb{Q}_p$. For $(x, y) \in G^2$, define the contraction $\mathrm{ctr}((x, y)) := x^{-1}y$.

4. For a prime ν of F, let $\mathfrak{g}_{\nu}$ denote the Lie algebra associated to $G_{F_{\nu}}$, which is an F_{ν}-vector space.

5. $\mathcal{K} = \prod_{\nu \in \Sigma_F} \mathcal{K}_{\nu} = \prod_{\upsilon \in \Sigma_{\mathbb{Q}}} \mathcal{K}_{\upsilon}$ a compact subgroup of $\mathbb{G}(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_F)$, where the factors at finite places are maximal at almost all places. When we have chosen G such that $G(F_{\infty})$ is compact, we often assume that $\mathcal{K}_{\infty} = \mathbb{G}(\mathbb{Q}_{\infty})$, however it may also be a maximal compact torus.

6. Let S be a finite set of finite places of $\mathbb{Q}$ such that $\mathbb{G}$ is split at each place in S. Almost always, this is chosen such that the class group

$$\mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A}_{\mathbb{Q}}^{S\infty})/\mathcal{K}^{S\infty}$$

is trivial, and so there is a canonical isomorphism

$$\mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A}_{\mathbb{Q}})/\mathcal{K} \xrightarrow{\sim} \mathbb{G}(\mathbb{Q}) \cap \mathcal{K} \setminus \mathbb{G}(\mathbb{Q}_S) =: X.$$

By the assumption that $\mathbb{G}$ is split at each place $\upsilon \in S$, we see that the field F must split completely over υ, and

$$\mathbb{G}_{\mathbb{Q}_{\upsilon}} \cong \left(G^{spl}\right)^d.$$

We will often write equality here, where it is understood that we pick a bijection now, and stick with it throughout. We will write $\mathcal{G} := \mathbb{G}(\mathbb{Q}_S)$.

7. For each place $\nu \in \Sigma_F$ dividing S, pick a finite index subgroup, $A_\nu < G_\nu$, of a maximal F_ν-split torus and let $A = \prod_{\nu|S} A_\nu$. By the rank of A we mean $|S|\text{rank}(\mathbb{G})$. A acts on X on the right.

8. For a metric group, G, and $\epsilon > 0$, we denote by B_ϵ^G the open ball of radius ϵ centred at the identity of G. If G acts on a set X, and $x \in X$, then $B_\epsilon^G(x) := B_\epsilon^G \cdot x$.

Chapter 2

Equidistribution Problems and Tori

2.1 Problem Statements

Let G be a semi-simple algebraic group defined over a number field F. Let V be a finite dimensional F-linear representation[1] of G with no trivial summand (meaning no positive dimension summand on which G acts trivially). Define the semi-direct product,

$$P := G \ltimes V,$$

where $(g, u) \cdot (h, v) = (gh, u + gv)$. Choose a level structure, given by a compact subgroup $\mathcal{K} = \mathcal{K}_G \ltimes \mathcal{K}_V \subset P(\mathbb{A}_F)$ satisfying

$$\mathcal{K} = \prod_{\nu \in \Sigma_F} \mathcal{K}_\nu,$$

where $\Sigma_F := \{\text{places of } F\}$, for $\nu \nmid \infty$, $\mathcal{K}_\nu$ is an open compact subgroup (maximal at almost all places).

The main object of interest (although it will soon be left behind in favour of a S-adic, and eventually fully adelic, covering space) is the double quotient

$$X_{\mathcal{K}} := P(F) \setminus P(\mathbb{A}_F)/\mathcal{K}.$$

This comes with a natural measure, coming from the Haar measure on $P(\mathbb{A}_F)$. In the case that the quotient is discrete, the double coset of (g, v) is given a mass inversely proportional to the size of $G(F) \cap (g, v)\mathcal{K}(g, v)^{-1}$. We mention a few specific cases:

[1] In many cases of interest the representation V will be zero.

- Suppose $V = 0$. If $G(F_\infty)$ is compact, then $X_{\mathcal{K}}$ is a finite collection of points when $\mathcal{K}_\infty$ is chosen to be maximal. This case is very similar to the construction of the class group of an algebraic torus, and in fact the collection of points is often referred to as the class set of that semi-simple algebraic group. Otherwise, $X_{\mathcal{K}}$ is a compact (possibly disconnected) real manifold.
  - For $G = \mathbb{PH}^\times$, the projective Hamiltonians over $\mathbb{Q}$, $X_{\mathcal{K}}$ will be a disjoint union of (possibly finite quotients of) the unit sphere S^2 if $\mathcal{K}_\infty$ is a maximal compact *torus*. If $\mathcal{K}_\infty = G(\mathbb{R})$, we get the finite set of points indexing the connected components of the previous space.
  - For the projective group of units of an arbitrary definite quaternion algebra, the above point is replaced by the surfaces of constant norm for the quadratic form associated to the quaternion algebra (e.g. ellipsoids over $\mathbb{R}$ for definite quaternion algebras over $\mathbb{Q}$).
- If $V = 0$ but $G(F_\infty)$ is no longer assumed to be compact, then $X_{\mathcal{K}}$ is a finite collection of real manifolds.
  - When $G = \mathrm{SL}_2(\mathbb{Q})$, we get the modular curve, a quotient of the upper half plane by a congruence subgroup. This is a moduli space of elliptic curves.
  - For $G = \mathrm{SL}_2(F)$ for F a totally real field, we get a Hilbert modular curve. This is a moduli space of abelian varieties with real multiplication.
- When $V \neq 0$, we get a space which fibres over the corresponding space with $V = 0$, and the fibres are real tori (quotients $\mathbb{R}^n/\mathcal{L}$ for a lattice $\mathcal{L} \subset \mathbb{R}^n$).
  - For example, the Kuga-Sato variety corresponds to $G = \mathrm{SL}_2$ and $V = \mathbb{G}_a^2$ with the natural left action of SL_2. The space $X_{\mathcal{K}}$ corresponds to a moduli space of elliptic curves (this is the related space with $V = 0$) *along with* a complex point of that elliptic curve (which corresponds to a point on the real torus mentioned above). This is also known as the universal elliptic curve.

The equidistribution problem that concerns us is related to anisotropic maximal tori contained in G. Let $T_i \leq G$ be a sequence of anisotropic tori in G with maximal rank, and $\xi_i = (g_i, v_i) \in P(\mathbb{A}_F)$ be a sequence of adelic points of P. Then the image of the composition

$$T_i(\mathbb{A}_F)\xi_i \hookrightarrow P(\mathbb{A}_F) \to X_{\mathcal{K}}$$

is a subset of $X_{\mathcal{K}}$. For a subgroup $\mathcal{T}_i \leq T_i(\mathbb{A}_F)$, we denote by $[\![\mathcal{T}_i \xi_i]\!]$ the image of $\mathcal{T}_i \xi_i$ in $X_{\mathcal{K}}$.

Question 2.1.1. *Under what conditions does a sequence $([\![\mathcal{T}_i \xi_i]\!])_{i=1}^{\infty} \subset X_{\mathcal{K}}$ equidistribute over $X_{\mathcal{K}}$?*

This is the first question which is interesting in the problem of equidistribution of torus orbits - we will call this the question of *single equidistribution*. Often, when $V \neq 0$, we will refer to this as the *Kuga-Sato case*. There is a second, related question, which we call *joint equidistribution*. Briefly, we note that the reason we take anisotropic tori is that for a torus to have a finite invariant measure on $T(F) \backslash T(\mathbb{A}_F)$, it must be that T has no F-characters (by Theorem 5.5 of [PR94] for example), and so is anisotropic. Therefore the only possible invariant probability measures on tori come from the anisotropic ones.

In dynamics and ergodic theory, a natural second question to consider after an equidistribution statement is a mixing statement. Equidistribution tells us that the orbits in question spread out to cover the space, however the action may not separate nearby points (therefore not mix together far away points, hence the name *mixing*). This dynamical consideration led Michel and Venkatesh in their ICM talk [MV06] to ask about the mixing properties in this arithmetic setting. This is what we call the joint equidistribution question below. In this setting, we consider the product $P \times P$, which contains the diagonally embedded torus $T_i^{\Delta} \subset P \times P$.

Question 2.1.2. *Take a sequence $([\![\mathcal{T}_i \xi_i]\!])_{i=1}^{\infty}$ as before, and for each i, an element $s_i \in T_i(\mathbb{A}_F)$. Under what conditions does the sequence $\left([\![\mathcal{T}_i^{\Delta}(\xi_i, s_i \xi_i)]\!]\right)_{i=1}^{\infty}$ equidistribute in $X_{\mathcal{K}} \times X_{\mathcal{K}}$?*

It may seem that when $V \neq 0$, the single equidistribution case looks more like the joint equidistribution case, however we have divided them like this due to a crucially important difference. In the single equidistribution case, the torus acting is maximal in the ambient group P, however in the joint equidistribution case the torus has half-maximal rank in $P \times P$!

Now, we wish to show how these questions can, in some cases, be seen as questions related to toric orbits on quotients of S-adic spaces. Moving between adelic statements and S-adic statements is a common procedure, so we will not spell out all the details.

The issue is that $X_{\mathcal{K}}$ doesn't have a well-defined action on it by the group $\mathcal{T}_i$. A common strategy to fix such a problem is to find a covering space of $X_{\mathcal{K}}$ and a subgroup of $\mathcal{T}_i$ such that there is an action of the subgroup on the covering space. To do this, we remove the factors of the maximal compact that lie in specific places.

First, to apply results that we wish to use later on, we will replace all the groups by their restriction of scalars to $\mathbb{Q}$. Thus, we define

$$\begin{aligned}
\mathbb{G} &:= \mathrm{Res}_{F/\mathbb{Q}} G \\
\mathbb{V} &:= \mathrm{Res}_{F/\mathbb{Q}} V \\
\mathbb{P} &:= \mathbb{G} \ltimes \mathbb{V} \\
\mathbb{T}_i &:= \mathrm{Res}_{F/\mathbb{Q}} T_i.
\end{aligned}$$

Also, since $\mathbb{G}(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_F)$, and likewise other formulae, we can consider $\mathcal{K} \subset \mathbb{G}(\mathbb{A}_{\mathbb{Q}})$ which is now a product over the places of $\mathbb{Q}$. We now pick a finite collection of finite places S such that $\mathbb{G}$ splits over S, or equivalently, G splits at all places of F lying above S. Furthermore, we choose for convenience a set of places large enough such that

$$\left|\mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A}_{\mathbb{Q}}^{S\infty})/\mathcal{K}^{S\infty}\right| = 1,$$

which can always be done (by finiteness of the class set for semi-simple groups). The consequence of this is that we have maps

$$\mathbb{G}(\mathbb{Q}) \cap \mathcal{K}^S \setminus \mathbb{G}(\mathbb{Q}_{S\infty})/\mathcal{K}_\infty \xrightarrow{\sim} \mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A}_{\mathbb{Q}})/\mathcal{K}^S.$$

In fact this can be easily upgraded to the group $\mathcal{P}$ using strong approximation on $\mathbb{G}_a$. That is, we get

$$X_{\mathcal{K},S} := \mathbb{P}(\mathbb{Q}) \cap \mathcal{K}^S \setminus \mathbb{P}(\mathbb{Q}_{S\infty})/\mathcal{K}_\infty \xrightarrow{\sim} \mathbb{P}(\mathbb{Q}) \setminus \mathbb{P}(\mathbb{A}_{\mathbb{Q}})/\mathcal{K}^S \twoheadrightarrow X_{\mathcal{K}}.$$

The advantage of removing the S-adic places of $\mathcal{K}$ is that now $\mathbb{P}(\mathbb{Q}_S)$ acts on the right of the larger space (this is in fact the crucial starting point for the entire ergodic method - we are now in the world on dynamics). Now let's consider the sets $[\![\mathcal{T}_i\xi_i]\!] \subset X_{\mathcal{K}}$. We can still consider the subgroup $\mathcal{T}_i \subset \mathbb{T}_i(\mathbb{A}_{\mathbb{Q}})$ and consequently the image of $\mathcal{T}_i\xi_i$ inside $\mathbb{P}(\mathbb{Q}) \setminus \mathbb{P}(\mathbb{A}_{\mathbb{Q}})/\mathcal{K}^S$, which we now call $[\mathcal{T}_i\xi_i]_S$, which will inherit a right action $\xi_{i,S}^{-1}\left(\mathcal{T}_i \cap \mathbb{T}_i(\mathbb{Q}_S)\right)\xi_{i,S}$. Recall that $\xi_{i,S} = \prod_{\nu \in S} \xi_{i,\nu}$ is the S-adic component of ξ_i.

To connect with homogeneous dynamics, we assume that all of the $\mathbb{T}_i$ are split at the places of S, and that $\xi_{i,S}^{-1}\left(\mathcal{T}_i \cap \mathbb{T}_i(\mathbb{Q}_S)\right)\xi_{i,S}$ is a finite index subgroup of a fixed maximal split torus. Consequently, the sets $[\mathcal{T}_i\xi_i]_S$ on $X_{\mathcal{K},S}$ are 'packets' of a finite number of orbits of a maximal split S-adic torus.

Question 2.1.3. *Under what conditions does a sequence* $([\mathcal{T}_i\xi_i]_S)_{i=1}^{\infty} \subset X_{\mathcal{K},S}$ *equidistribute over* $X_{\mathcal{K},S}$*?*

This is now a question of orbits of tori on quotients of S-adic groups by lattices. In fact, we will see, via a standard procedure, in Section 3.3.2, that we can even consider the fully adelic question on $[\mathbb{G}(\mathbb{A})] := \mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A})$ and $[\mathbb{P}(\mathbb{A})] := \mathbb{P}(\mathbb{Q}) \setminus \mathbb{P}(\mathbb{A})$. On this space, we again have an image of $\mathcal{T}_i\xi_i$ which we denote $[\mathcal{T}_i\xi_i]$.

2.2 Specific Cases and their Tori

The cases of interest to us in this thesis are the following:

Case I: Quaternion Algebras

We consider the equidistribution of toric orbits in definite quaternion algebras over totally real fields. Let B/F be a quaternion algebra over a totally real field such that for every real place $\nu \in \Sigma_{F,\infty}$,

$$B \otimes_F F_\nu \cong \mathbb{H},$$

where $\mathbb{H}$ is the real non-split quaternion algebra (Hamilton's quaternions). In this case, in the notation of Section 2.1, we let $G = B^\times/F^\times = PB^\times$ be the algebraic group over F corresponding to the projective group of units of B. The set of real points of this group is compact. We will not prove anything concerning this case, since equidistribution in the single quaternion algebra case is known over an arbitrary number field (see [Ein+07, Theorem 4.6]), however we will discuss the ergodic method of proving such a theorem and the obstructions to doing this in more general cases.

Case II: Joint Equidistribution

We also consider the joint equidistribution problem in the setting above. The most significant contribution of this thesis is to extend the result of [Kha17] for quaternion algebras over $\mathbb{Q}$ to the case of totally real fields.

Case III: Kuga-Sato Varieties

In this setting, we again work over a totally real field. We will often consider the completely general case here, where possible. However we will also often set $G = \mathrm{SL}_{2,F}$ and $V = \mathbb{G}_{a,F}^2$ with the usual action of G on it. This is to generalise [Kha19b].

Case IV: Unitary Groups

Another case of interest, although one where we will be able to achieve significantly less, is the case of definite unitary groups over totally real fields. Eventually, we are interested in both the single and joint equidistribution questions in this setting.

Let F be a totally real algebraic number field, and let (K, τ) be a CM type consisting of a CM field K quadratic over F and a complex representation τ of K such that $\tau \oplus \overline{\tau}$ is isomorphic to the regular representation of $K/\mathbb{Q}$. Thus τ consists of a collection $\{\tau_\sigma\}_{\sigma \in \Sigma_{F,\infty}}$ where τ_σ is a complex embedding of K extending $\sigma : F \to \mathbb{R}$. Let $r \geq 2$, and choose a positive definite Hermitian space $(V, \langle \cdot, \cdot \rangle)$ of dimension r over K. After choosing a basis, this corresponds to a Hermitian matrix $J \in \mathrm{GL}_r(K)$ such that for each $\sigma \in \Sigma_{F,\infty}$, $\tau_\sigma(J) \in \mathrm{GL}_r(\mathbb{C})$ is positive definite.

Then we get algebraic groups over F defined by

$$\begin{aligned} \widetilde{G}(R) &= \left\{\alpha \in \mathrm{GL}_{K \otimes_F R}(V) | \langle \alpha v, \alpha w \rangle = \nu(\alpha) \langle v, w \rangle, \forall v, w \in V, \text{ with } \nu(\alpha) \in R^\times \right\} \\ G(R) &= \left\{\alpha \in \widetilde{G}(R) | \nu(\alpha) = 1\right\} \\ PG(R) &= \widetilde{G}(R)/(K \otimes_F R)^\times \end{aligned}$$

Respectively, these have rank $r+1, r, r-1$ over F. They are considered as closed subgroups of $\mathrm{Res}_{K/F}\mathrm{GL}_{r,K} \times \mathrm{GL}_{1,F}$, $\mathrm{Res}_{K/F}\mathrm{GL}_{r,K}$, and $\mathrm{Res}_{K/F}\mathrm{PGL}_{r,K}$ respectively via the chosen basis, using the conjugate transpose $\alpha \mapsto \alpha^\dagger$ using the complex conjugation on K/F (often we will lazily drop the restriction of scalars here and state for example that $G \leq \mathrm{GL}_{r,K}$). Indeed, we find that

$$\begin{aligned} \widetilde{G}(F) &= \left\{\alpha \in \mathrm{GL}_r(K) : \alpha^\dagger J \alpha = \nu(\alpha) J, \text{ with } \nu(\alpha) \in F^\times \right\} \\ G(F) &= \left\{\alpha \in \mathrm{GL}_r(K) : \alpha^\dagger J \alpha = J\right\} \\ PG(F) &= \widetilde{G}(F)/K^\times \end{aligned}$$

An important property of the unitary group defined above is that it splits over the field K. Indeed, for any $R \supset K$,

$$\mathrm{GL}_{K \otimes_F R}(V \otimes_F R) \xrightarrow{\sim} \mathrm{GL}_R(V \otimes_K R) \times \mathrm{GL}_R(V \otimes_K R)$$

Under this isomorphism, the image of $\widetilde{G}(R)$ is

$$\left\{(A, B) : B^T J A = \nu J, A^T \overline{J} B = \nu \overline{J}\right\}.$$

Since A, ν uniquely determine B, we see that projection onto the first factor, along with recording the value of ν, gives an isomorphism

$$\widetilde{G}_K \xrightarrow{\sim} \mathrm{GL}_{r,K} \times \mathrm{GL}_{1,K}.$$

Furthermore, the involution of $\mathrm{GL}_{r,K} \times \mathrm{GL}_{1,K}$ which fixes $\widetilde{G}(F) \subset \widetilde{G}(K) = \widetilde{G}_K(K)$ is given by

$$(A, \nu) \mapsto (\overline{\nu} J^{-1} (A^\dagger)^{-1} J, \overline{\nu}).$$

2.2.1 Maximal Tori

As mentioned in Section 2.1, we are interested in the anisotropic maximal tori contained in G. In the study of algebraic tori, the following result is indispensable.

Theorem 2.2.1 ([Ono61])**.** *There is an equivalence of categories between tori defined over F and split over L (for L a finite Galois extension of F) and the category of finitely generated $\mathbb{Z}$-torsion-free $\mathbb{Z}[\mathrm{Gal}(L/F)]$-modules. This equivalence sends $T \mapsto \widehat{T}$ where $\widehat{T}$ is the abelian group of characters of T defined over L.*

In particular, the $\mathbb{Z}$-rank of the Galois module, $\widehat{T}$, in Theorem 2.2.1 corresponds to the dimension of the torus T. Therefore, in Cases I-III, since the tori we consider are dimension 1, they correspond to one of two cases:

1. The module $\mathbb{Z}$ with the trivial Galois action, so $T \cong \mathbb{G}_{m,F}$. However this is not anisotropic.

2. The module $\mathbb{Z}$ with a non-trivial action of some $\mathrm{Gal}(E/F)$. Since any non-trivial action is simply surjective onto $\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, we may assume that E is a CM quadratic extension of F (it must be CM so that the points at infinity are compact), and

$$T \cong \mathrm{Res}_{E/F}(\mathbb{G}_{m,E}) / \mathbb{G}_{m,F}, \tag{2.1}$$

   where Res refers to the restriction of scalars. If, instead of the adjoint (projective) group, we use the simply connected group of norm one elements in B, then the tori will be $\mathrm{Res}^1 \mathbb{G}_{m,E}$ where Res^1 denotes the norm 1 subgroup.

The first case is not anisotropic (as we require), and so can be discarded, and therefore all tori that we consider in (the projective versions of) Cases I-III are of the form (2.1).

In fact, we can say something stronger:

Proposition 2.2.2. *Let T be an anisotropic maximal F-rational algebraic torus in any of $B^\times$, $B^{(1)}$ or $PB^\times$. Then there is a CM extension E/F and an F-algebra embedding $f : E \hookrightarrow B$ such that f induces an isomorphism of T with $E^\times, E^1$ or $E^\times/F^\times$ respectively.*

Proof. Let $\widetilde{T}$ be either $T, F^\times T$, or $\pi^{-1}(T)$ respectively (where $\pi : B^\times \to PB^\times$ is the quotient by scalars map). Then $\widetilde{T} \leq B^\times$ is a maximal torus. Let $C = C_B(\widetilde{T})$ be the F-subalgebra of B centralising $\widetilde{T}$. By extending to $\overline{F}$, we see that $\widetilde{T}$ is conjugate to the diagonal torus in $M_2(\overline{F})$, and so $C_{\overline{F}} \cong \overline{F} \times \overline{F}$, and C/F is a 2-dimensional etale algebra. Therefore $C = E$ for a quadratic extension E/F, or $C = F \times F$. In the latter case, however, T is F-split. □

The tori in Case IV are slightly harder to identify, but the same approach works.

Proposition 2.2.3. *The anisotropic tori in PG are all constructed by taking a product of CM fields $M = \bigoplus_{i=1}^s M_i$ each containing K with total degree $[M : K] = r$ and a K-linear embedding $M \to \mathrm{End}_K(V)$. The corresponding torus is isomorphic to*

$$\mathrm{Ker}\left(N_{M/M^+} : \mathrm{Res}_{M/F}\mathbb{G}_{m,M} \to \mathrm{Res}_{M^+/F}\mathbb{G}_{m,M^+}/\mathbb{G}_m\right)/\mathrm{Res}_{K/F}\mathbb{G}_m,$$

where $M^+ = \prod_{i=1}^s M_i^+$ is the product of the maximal totally real subfields $M_i^+ \leq M_i$. Likewise in the case of $\widetilde{G}$ the tori are

$$\mathrm{Ker}\left(N_{M/M^+} : \mathrm{Res}_{M/F}\mathbb{G}_{m,M} \to \mathrm{Res}_{M^+/F}\mathbb{G}_{m,M^+}/\mathbb{G}_m\right)$$

and in G the tori are

$$\mathrm{Ker}\left(N_{M/M^+} : \mathrm{Res}_{M/F}\mathbb{G}_{m,M} \to \mathrm{Res}_{M^+/F}\mathbb{G}_{m,M^+}\right).$$

Proof. Consider $T \subset PG$, and its preimage $\widetilde{T} \subset \widetilde{G}$. Then since we have[2] $\widetilde{G} \subset \mathrm{GL}_{r,K} \times \mathrm{GL}_{1,K} \subset \mathrm{End}_K(V) \times K$, we can consider $C := C_{\mathrm{End}_K(V)}\left(\pi_1\widetilde{T}\right)$ (where π_1 is the projection onto the first factor) which is a K-subalgebra of $\mathrm{End}_K(V)$. By extending scalars to $\overline{K}$, the algebraic closure of K, we can see that $\pi_1(\widetilde{T}_{\overline{K}})$ can be written as the set of diagonal matrices embedded diagonally into the product

$$\mathrm{End}_{\overline{K}}(V_{\overline{K}}) \times \mathrm{End}_{\overline{K}}(V_{\overline{K}})$$

and therefore $C \otimes_F \overline{K} \cong \overline{K}^{2r}$. Thus C is a K-algebra that is etale over F. Furthermore, it contains a copy of K which acts invertibly on C (the scalar matrices), so is an etale K-algebra. Therefore, we have

$$C \cong \prod_{i=1}^s M_i,$$

[2]As mentioned previously, we often drop restriction of scalars to ease readability: in fact $\widetilde{G} \leq \mathrm{Res}_{K/F}\mathrm{GL}_{r,K} \times \mathrm{Res}_{K/F}\mathrm{GL}_{1,K}$.

where M_i is a finite separable field extension of K for each $i = 1, ..., s$, and

$$\sum_{i=1}^{s} [M_i : K] = r.$$

Therefore,

$$T \subset \left(\prod_{i=1}^{s} \mathrm{Res}_{M_i/F} \mathbb{G}_{m,M_i} \right) / \mathrm{Res}_{K/F} \mathbb{G}_{m,K}. \tag{2.2}$$

The character lattice for $\prod_{i=1}^{s} \mathrm{Res}_{M_i/F} \mathbb{G}_{m,M_i}$ is given by

$$\Lambda := \bigoplus_{i=1}^{s} \mathbb{Z} \left[\mathrm{Hom}_F(M_i, \overline{F}) \right],$$

with the natural $\mathrm{Gal}\left(\overline{F}/F\right)$-action. This comes with a surjective evaluation map,

$$\mathrm{ev}_K : \Lambda \to \widehat{\mathrm{Res}_{K/F} \mathbb{G}_{m,K}}$$

given by the sum of the evaluation maps on each factor (sending $\sum_{\sigma: M_i \to \overline{F}} a_\sigma \sigma \mapsto \sum_\sigma a_\sigma \sigma|_K$). Therefore, equation (2.2) corresponds to the fact that $\widehat{T}$ is a quotient of $\ker(ev)$, with rank $r - 1$ and no real characters (since the real points of the torus are compact). Therefore, all the characters in $\ker(\mathrm{ev})$ which are defined over $\mathbb{R}$ must be in the kernel of the quotient map. These are

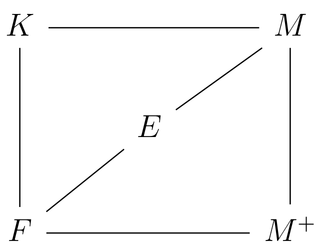
$$\ker(\mathrm{ev}) \cap \bigoplus_{i=1}^{s} \mathbb{Z} \left[\sigma + \overline{\sigma} : \sigma \in \mathrm{Hom}_F(L_i, \overline{F}) \right]$$

which has rank $r-1$. Since $\ker(\mathrm{ev})$ has rank $2r-2$, these must be precisely the kernel of the quotient map. It is easy to see that the resulting Galois representation is the character lattice of

$$\mathrm{Ker} \left(\bigoplus_{i=1}^{s} \mathrm{Res}_{M_i/F} \mathbb{G}_{m,M_i} \to \left(\bigoplus_{i=1}^{s} \mathrm{Res}_{M_i^+/F} \mathbb{G}_{m,M_i^+} \right) / \mathbb{G}_m \right) / \mathrm{Res}_{K/F} \mathbb{G}_m.$$

as required. □

This may not immediately match our expectation that the quaternion algebra case is simply the unitary case with $r = 2$, until we notice that in that case M will be Galois with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$ and we have a diagram of fields



where E is another quadratic CM extension of F, and then the inclusion $E \hookrightarrow M$ gives an isogeny of F-rational tori between $\mathbb{G}_{m,E}/\mathbb{G}_{m,F}$ as given previously for the quaternion algebra case and the one described in Proposition 2.2.3. This is a low-dimensional 'accidental isomorphism', and in fact we can see that that for $r \geq 3$ no such accident can occur.

Proposition 2.2.4. *Let T be a torus as in the previous Proposition, where we assume for convenience that M is a single CM field, Galois over K. Then when $r \geq 3$, the splitting field of T is M^{gal}, the Galois closure of M.*

Proof. If we fix an embedding $f_1 : M \to \overline{F}$, let $\{f_1, ..., f_r\}$ be the set of F-linear embeddings $M \to \overline{F}$ agreeing with f_1 on K. Then, we may check that the Galois representation on the dual lattice for T can be written explicitly as

$$\mathrm{Gal}(M^{gal}/F) \to \mathrm{Aut}\left(\left(\bigoplus_{1\leq i\leq r} \mathbb{Z}f_i\right)^{\mathrm{tr}=0}\right)$$

where the action is given by on a basis $\{f_i - f_1 : 2 \leq i \leq r\}$ by

$$\sigma \cdot (f_i - f_1) = (\sigma \circ f_i - f_1) - (\sigma \circ f_1 - f_1), \forall \sigma \in \mathrm{Gal}\,(M/K)$$
$$\tau \cdot (f_i - f_1) = -(f_i - f_1)$$

where τ is the complex conjugation in $\mathrm{Gal}(M^{gal}/F)$. We see that for $\sigma \in \mathrm{Gal}\left(M^{gal}/K\right)$ to act trivially, it must preserve f_i for all i, and so is trivial. If $\tau\sigma \in \mathrm{Gal}\left(M^{gal}/F\right)$ acts trivially, then it must be the case that $\sigma f_0 = f_i, \sigma f_i = f_0$ for all $2 \leq i \leq r$. Clearly this is only possible if $r = 2$. $\square$

We must also identify the possible rational linear representations of tori. This is because when we consider the Kuga-Sato setting of $\mathbb{P} = \mathbb{G} \ltimes \mathbb{V}$ with a maximal rational anisotropic torus $\mathbb{T} \leq \mathbb{G}$, this makes $\mathbb{V}$ into such a rational representation of the torus. When approaching correlations in this setting we will be able to decompose $\mathbb{V}$ into irreducible representations. In general, this is also useful for the identification of 'explicit coordinates' since, as explained in Section 2.4, these coordinates are really computed via a decomposition of the Lie algebra of H into representations of T. Recall that for the action of the Galois group, we use ${}^{\sigma}(\cdot)$. The following is well-known:

Proposition 2.2.5. *There is a bijection between the set of irreducible rational representations $\mathbb{V}$ of a torus $\mathbb{T}$ and the Galois orbits in $\widehat{\mathbb{T}}$.*

Proof. Given a representation $\mathbb{V}$ of $\mathbb{T}$, passing to $\overline{\mathbb{Q}}$-points we can diagonalise

$$\mathbb{V}\left(\overline{\mathbb{Q}}\right) = \bigoplus_{\chi \in \widehat{\mathbb{T}}} V_\chi$$

where for $x \in V_\chi$, $t \cdot x = \chi(t)x$. Therefore for any $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$,

$$t \cdot {}^\sigma x = {}^\sigma\left({}^{\sigma^{-1}}t \cdot x\right) = (\sigma \cdot \chi)(t)\, {}^\sigma(x),$$

i.e. $\sigma\left(V_\chi\right) = V_{\sigma \cdot \chi}$ where the action of σ on $\widehat{\mathbb{T}}$ is given by $\chi \mapsto \sigma \circ \chi \circ \sigma^{-1}$.

Conversely, given any Galois-fixed multiset, Σ, of points in $\widehat{\mathbb{T}}$, we take $V = \bigoplus_{\chi \in \Sigma} V_\chi$ with the natural $\mathbb{T}\left(\overline{\mathbb{Q}}\right)$-action, and define an alternative Galois action on this via

$$\sigma * (x, \chi) := \left({}^\sigma x, \sigma \cdot \chi\right).$$

A short computation verifies that this action satisfies

$$\sigma * (t \cdot x) = {}^\sigma t \cdot (\sigma * (x))$$

which implies that the fixed points of the $*$-Galois action give a $\mathbb{Q}$-rational representation of $\mathbb{T}$. These two processes are inverses to each other and clearly irreducible representations are sent to multisets consisting of a single Galois orbit with multiplicity 1. □

Clearly this result also applies to a different number field as the base field instead of $\mathbb{Q}$. Considering the analysis above, we have a few possible scenarios:

1. For $\mathbb{P} = \mathbb{G} \ltimes \mathbb{V}$ with $\mathbb{G} = \mathrm{ResSL}_{2,F}$ and $\mathbb{V}$ the restriction of scalars of an F-linear representation of $\mathrm{SL}_{2,F}$, we see that the anisotropic tori $\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E}$ with character group
$$\widehat{\mathbb{T}} = \mathbb{Z}$$
and non-trivial Galois action via $\mathrm{Gal}\,(E/F)$, we see that the Galois-orbits are $\{0\}, \{\chi^n \oplus \overline{\chi^n}\}$ for any $n \geq 1$. The first gives the trivial F-linear representation. The second gives the 2 dimensional F-linear representation of $\mathbb{T}$ on E via $t \cdot x = t^n x$. Thus $\mathbb{V}$ restricts to a sum of these irreducible $\mathbb{T}$-representations over F.

2. If we were to look at $\mathbb{P} = \mathbb{G} \ltimes \mathbb{V}$ with $G = PG$ as in case IV, the analysis is a little more complicated. Firstly, we can assume that the representation is not induced from a representation of the torus constructed in the same way on some CM subfield $L \leq M$ containing K. This reduces us to considering Galois orbits on $\widehat{\mathbb{T}}$ such that $\mathrm{Gal}\left(M^{gal}/K\right)$ acts faithfully.

Proposition 2.2.6. *Suppose that V is an irreducible F-linear representation of T, a torus in PG as above. Then, either V is the trivial representation, or there is a CM field $K \leq L \leq M^{gal}$ such that $V(F) = L$, and for some $a_i \in \mathbb{Z}$, the action of T on F points is given by*

$$m \cdot x = \prod_{1 \leq i \leq r} f_i(m)^{a_i} x \in L$$

where $\{f_i : 1 \leq i \leq r\} = \mathrm{Hom}_F(M, \overline{F})$, and $\sum_i a_i = 0$.

Proof. Let

$$v = \sum_{1 \leq i \leq r} a_i f_i$$

satisfy $\sum_i a_i = 0$. Let $f : \mathrm{Gal}\left(M^{gal}/F\right) \to \mathrm{Aut}(\bigoplus_{1 \leq i \leq r} \mathbb{Z} f_i^{tr=0})$. We will write $\sigma(i)$ for the index of $\sigma \circ f_i$ for any $\sigma \in \mathrm{Gal}\left(M^{gal}/K\right)$. First note that unless $v = 0$, the complex conjugation τ is never in the stabiliser of v.

As a simple case, if the stabiliser of v in $\mathrm{Gal}(M^{gal}/F)$ is trivial, then the corresponding irreducible representation is given on F-points by the action of $M^{N \in F}/K^{\times}$ on M^{gal} via

$$m \cdot x = \prod_i f_i(m)^{a_i} x.$$

This is well-defined since for $\zeta \in K^{\times}$, the product of embeddings $\prod_i f_i(k)$ is identically 1.

Consider the extension L/F contained in M^{gal} corresponding to $\mathrm{Stab}(v) \leq \mathrm{Gal}\left(M^{gal}/K\right)$. We divide into two possible cases.

Firstly, suppose that $K \leq L$, then by the definition of the Galois action on the character lattice, we see that

$$\prod_i f_i(m)^{a_i} \in L,$$

and therefore the representation of the torus is given by the F-vector space L with the action of m as above.

Secondly, suppose that $K \not\leq L$. We know that $\tau|_L \neq 1$, i.e. L is not totally real. Then for $\sigma \in \mathrm{Gal}\left(M^{gal}/L\right)$,

$$^{\sigma}\left(\prod_i f_i(m)^{a_i}\right) = \begin{cases} \prod_i f_i(m)^{a_i}, & \text{if } \sigma|_K = 1, \\ ^{\tau}\left(\prod_i f_i(m)^{a_i}\right), & \text{o/w }. \end{cases}$$

This implies that

$$\prod_i f_i(m)^{a_i} \in KL^+$$

and therefore the representation is given by KL^+. $\square$

2.3 Open Compact Subgroups and Associated Orders

In order to normalise the following results (e.g. normalise integrals correctly etc), we need to choose suitable open compact subgroups inside $G(\mathbb{A}_F^\infty)$, and compact identity neighbourhoods at the infinite places.

2.3.1 Quaternionic Case

Recall that we have a quaternion algebra B/F (which we treat as a functor $B(R) = B \otimes_F R$ for an F-algebra R). Let $\mathcal{O}$ be a maximal $\mathcal{O}_F$-order in the rational points $B(F)$, and we formed the F-algebraic group $G = PB^\times$. Then we define compact open subgroups $\mathcal{K}_\nu < G(F_\nu)$ for each $\nu \in \Sigma_F^\infty$ as the image of $\mathcal{O}_\nu^\times$ in $G(F_\nu)$, where $\mathcal{O}_\nu = \mathcal{O} \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\nu}$ is the ν-adic local order in $B(F_\nu)$ associated to $\mathcal{O}$ by localisation.

We also fix a maximal compact torus $\mathcal{K}_\nu < G(F_\nu)$ for each $\nu|\infty$. At each place, we can choose an isomorphism

$$B_\nu = \mathbb{H} \text{ or } M_2(F_\nu),$$

and in the latter case we can furthermore assume that the compact torus $\mathcal{K}_\nu$ corresponds to $PSO_2(F_\nu)$. If $B_\nu = \mathbb{H}$, then define the compact subset

$$\mathcal{O}_\nu = \{x \in \mathbb{H} : \mathrm{Nr}(x) \leq 1\} \subset B_\nu$$

Then $\mathcal{O}_\nu^\times = \mathbb{H}^1$, which is compact, but not an identity neighbourhood. Thus we also consider a thickening of this given by

$$\widetilde{\Omega_\nu} := \left\{ x \in \mathbb{H}^\times : \frac{1}{4} \leq \mathrm{Nr}(x) \leq 4 \right\}.$$

If $B_\nu = M_2(F_\nu)$, we define analogous objects using a different norm. Let $||A||_\infty = \sup_{0 \neq v \in F_\nu^2} \frac{|Av|}{|v|}$ be the operator norm, and then define

$$\begin{aligned} \mathcal{O}_\nu &:= \{A : ||A||_\nu \leq 1\} \\ \mathcal{O}_\nu^\times &:= \{A : ||A||_\nu = 1, \det(A) > 0\} = \mathrm{SO}_2(F_\nu) \\ \widetilde{\Omega_\nu} &:= \left\{A : |\det(A)| > 0, ||A^{\pm 1}||_\nu \leq 2\right\}. \end{aligned}$$

Notice that at the infinite places, when B_ν is split $\dim_F \mathcal{O}_\nu^\times = 1$, whereas when B_ν is non-split $\dim_F \mathcal{O}_\nu^\times = 3$.

Now, we can associate to a homogeneous toral set (in either the simply connected or adjoint case) an order in the following way. Let $[\mathcal{T}g]$ be a homogeneous toral set, where $\mathcal{T} \leq T(\mathbb{A}_F)$ is a finite index subgroup for $T \leq G$ an anisotropic maximal torus, and $g \in G(\mathbb{A}_F)$. In Section 2.2.1, we associated to such a $T \leq G$ an F-algebra embedding $E \to B$, and we identify E with its image under this embedding.

Definition 2.3.1. *For each place ν of F, let*

$$\Lambda_\nu = E_\nu \cap g_\nu \mathcal{O}_\nu g_\nu^{-1}.$$

For almost all ν this is equal to $E_\nu \cap \mathcal{O}_\nu$ which comes from the global embedding $E \to B$. Therefore for almost all such ν, $\Lambda_\nu = (E \cap \mathcal{O})_\nu \subset E_\nu$ is the maximal order. Therefore, we can make the definition

$$\Lambda := \bigcap_{\nu \nmid \infty} \Lambda_\nu \subset E$$

which is an $\mathcal{O}_F$-order of E.

2.3.2 Unitary Case

Similarly, consider a maximal $\mathcal{O}_K$-order $\mathcal{O} \subset \mathrm{End}_K(V)$ which is preserved by the involution defining G. Then for each $\mathfrak{p}$ we can take the localisation $\mathcal{O}_\mathfrak{p} \subset \mathrm{End}_K(V)$, and therefore we get the open compact subgroup $\mathcal{O}_\mathfrak{p}^\times \subset \mathrm{Aut}_K(V)$. This gives open compacts $\mathcal{K}$ for each of $\widetilde{G}(\mathbb{A}_F^\infty), G(\mathbb{A}_F^\infty)$ and $PG(\mathbb{A}_F^\infty)$.

In this case, there are more possibilities for the place at infinity, since the unitary group $G_\nu \cong U(p_\nu, q_\nu)$ can be any signature (with $p_\nu + q_\nu = r$). However, in this case the forced generality actually adds uniformity (which was there in the quaternion case but not transparent). All the real unitary groups $U(p,q)$ embed in $M_r(\mathbb{C})$, and we simply use the corresponding operator norm $||\cdot||_\nu$ on $M_r(\mathbb{C})$, and give the same definitions as the split case of the quaternion algebra setting. Since any two embeddings of $U(p,q)$ into $M_r(\mathbb{C})$ are $GL_r(\mathbb{C})$-conjugate, these definitions are independent of our choices.[3]

Why is this a sensible choice to make? The answer comes from the Cartan decomposition, $G = KAK$ of a Lie group. We can view this as recording the eigenvalues of a matrix within the diagonal subgroup A. To choose a nice compact neighbourhood of the identity in G it suffices to do the same in A. Bounding the operator norm and

[3]To see this, note that an outer form is determined by a matrix $\theta \in \mathrm{GL}_r(\mathbb{C})$ such that $\theta\rho(\overline{\theta}) = 1$ where ρ is the outer automorphism of $\mathrm{GL}_r(\mathbb{C})$ given by conjugation by the anti-diagonal. The corresponding subgroup of $\mathrm{GL}_r(\mathbb{C})$ is the fixed subgroup under $A \mapsto (\mathrm{Ad}(\theta) \circ \rho)\,(\overline{A})$. Any other isomorphic embedding corresponds to another θ' such that the cocycles differ by a coboundary. Unwinding the definitions, this is iff there is a ϑ such that $\theta = \overline{\vartheta}\theta'\rho(\vartheta)^{-1}$. In this case, we see that $G'(\mathbb{R}) = \mathrm{Ad}\left(\overline{\vartheta}\right)^{-1} G(\mathbb{R})$.

its inverse corresponds to giving a neighbourhood of 1 in which all the eigenvalues must lie. This gives a compact neighbourhood of the torus A, and therefore one of G.

Let us check that this recovers the definition given with the reduced norm in the non-split case of the quaternion algebra. We have an embedding

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathrm{M}_2(\mathbb{C}) \\ z = a + ib + jc + kd &\longmapsto \begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix} \end{aligned}$$

which is an $\mathbb{R}$-algebra injection (here we have used the standard basis $\{1, i, j, k\}$ of the Hamilton quaternions). The characteristic equation of this matrix is $\lambda^2 - \mathrm{Tr}(z)\lambda + \mathrm{Nr}(z) = 0$ and so we see that both eigenvalues have the same absolute value equal to $\sqrt{\mathrm{Nr}(z)}$. Therefore this is the operator norm on the image of the above embedding, and we see that the two cases match up.

We can also define the orders Λ_ν, Λ of M_ν and M analogously. Note an important difference: in the quaternion algebra case, since we have a lucky isomorphism as explained in Section 2.2.1, we get an order in E which has dimension $r = 2$ over F, whereas in the unitary case we can only get an order in M which has dimension $2r$ over F. However, the presence of $K \subset M$ and the fact that Λ is an $\mathcal{O}_K$-order preserved by complex conjugation (due to the fact that $\mathcal{O}$ is preserved by the involution on $M_r(K)$) means that practically we still have an order in an r-dimensional space, M^+.

2.4 Coordinates With Respect to Maximal Tori

This section is modelled on Section 5 of [Kha17], which discusses the coordinate representations of quaternion algebras relative to embedded quadratic imaginary fields. We briefly recall the results in the quaternionic case (which generalise immediately to totally real fields - we will simply give sketch proofs, since more careful proofs are given in the unitary case) and discuss similar representations in the unitary case.

There is actually a very intuitive phenomenon occurring here: given a non-split maximal torus T inside a group G, we may take a Galois extension so that the torus splits. Then, following the normal theory of Lie groups we can attach a root system and decompose any representation into weights. Since all of this is happening over a suitable extension, the Galois group of that extension acts on these objects. Suppose G acts on a linear algebraic group H, then the Galois group acts on the weights arising in the representation of G on $\mathrm{Lie}(H)$, and the representation of T on $\mathrm{Lie}(H)$

decomposes into a sum over the orbits of this Galois action, as in Proposition 2.2.5. Suppose that we wish to find a nice description for the adjoint action of T on H, then it is natural to consider this decomposition of the Lie algebra, and hope that for sufficiently integral elements of H, we can lift them to integral elements of $\mathrm{Lie}(H)$.

When considering correlations, as we will do later, it is vital to understand carefully the object

$$G \setminus (G \ltimes H)/T.$$

Naturally this relates to the adjoint action of T on H, and so the above construction becomes useful.

2.4.1 Quaternionic Case

Given an anisotropic maximal torus $T \subset G = PB^\times$ where B is a quaternion algebra over a totally real field F, we know that $T \cong E^\times/F^\times$ for a CM extension E/F. Furthermore G splits over the extension E, so there is an isomorphism $G_E \cong \mathrm{PGL}_{2,E}$. In fact, we have an isomorphism $B_E \cong M_{2,E}$. B must split over E since $T_E(E) \cong (E \otimes_F E)^\times/E^\times \cong E^\times$ is a split maximal torus.

The result of Section 5 in [Kha17] is to choose carefully an isomorphism $B_E \cong M_{2,E}$ such that the F-rational points $B(F) \subset M_2(E)$ coming via base change can be easily identified. Recall once again that the Galois action of $\sigma \in \mathrm{Gal}\left(\overline{F}/F\right)$ is denoted by $x \mapsto {}^\sigma x$.

Proposition 2.4.1. *There is an isomorphism $B_E \cong M_{2,E}$ such that for some element $\epsilon \in F^\times$ and any field $M \in \{F, F_\nu : \nu \in \Sigma_F\}$, the M-rational points of B in $M_{2,E}(M \otimes_F E)$ are given by*

$$B(M) = \left\{ \begin{pmatrix} a & \epsilon b \\ {}^\sigma b & {}^\sigma a \end{pmatrix} : a, b \in M \otimes_F E \right\}.$$

Furthermore, B splits over M if and only if $\epsilon \in N_{M\otimes_F E/M}\left((M \otimes_F E)^\times\right)$. If this is the case, i.e. $\epsilon = {}^\sigma f f$, then by replacing b with $b/{}^\sigma f$, we can restate this as

$$B(M) = \left\{ \begin{pmatrix} a & bf \\ {}^\sigma b/f & {}^\sigma a \end{pmatrix} : a, b \in M \otimes_F E \right\}.$$

Sketch. This follows from examining the cohomology group $\mathrm{H}^1(E/F, \mathrm{Aut}\,(\mathrm{GL}_{2,E}))$ which defines the F-forms of $\mathrm{GL}_{2,F}$ that split over E. Since there are no diagram automorphisms (i.e. symmetries of the Dynkin diagram) for A_1, the outer automorphism group is trivial, and the full automorphism group is given by

$$\mathrm{Aut}\,(\mathrm{GL}_{2,E}) = \mathrm{PGL}_2(E).$$

Since T_E is a split maximal torus, we can arrange for this to be the diagonal matrices, but the F-rational points $T(F)$ cannot be the diagonal F-rational matrices (since B is not split). Any element of $c \in H^1(E/F, \mathrm{Aut}(\mathrm{GL}_{2,E}))$ is determined by the automorphism $c(\tau)$ and if this additionally preserves the diagonal matrices but acts non-trivially on them, it can be represented by an element $\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$ with $\epsilon \in F^\times$. We can then unwind the definition of the cocycle to show that the F-rational points of the group G corresponding to a cocycle $c : \mathrm{Gal}\,(E/F) \to \mathrm{PGL}_{2,E}$ are the solutions to

$$A = \mathrm{Ad}(c(\tau))\overline{A}.$$

Where both τ and the overline correspond to complex conjugation. □

We showed in Section 2.3 how to attach an order $\Lambda \subset E$ to a maximal order $\mathcal{O} \subset B(F)$ for any homogeneous toral set. Now, we see that this process is almost reversible, that is we can *almost* identify the local order $g_\nu^{-1}\mathcal{O}_\nu g_\nu$ in terms of Λ_ν. For this, we require the inverse different ideal

$$\widehat{\Lambda_\nu} = \{a \in E_\nu | \mathrm{Tr}(a\Lambda_\nu) \subset \mathcal{O}_{F_\nu}\}.$$

This is a proper fractional Λ_ν-ideal with inverse $\mathfrak{d}_{\Lambda_\nu/F_\nu} \subset \Lambda_\nu$.

Proposition 2.4.2. *For any finite place ν there is some $\tau_\nu \in E_\nu^\times$ such that*

$$g_\nu \mathcal{O}_\nu g_\nu^{-1} \subset \left\{ \begin{pmatrix} \alpha & \beta \upsilon_\nu \tau_\nu \\ {}^\sigma\beta/\tau_\nu & {}^\sigma\alpha \end{pmatrix} : \alpha, \beta \in \widehat{\Lambda_\nu} \right\}.$$

If B is split at ν then $\upsilon_\nu = 1$, if B is ramified and E is inert at ν then υ_ν is a uniformiser in $\mathcal{O}_{F_\nu}$, and if both B, E are ramified at ν then υ_ν is a unit which is not a $E_\nu^\times$-norm. Also, $\tau_\nu \in \Lambda_\nu^\times$ for almost all ν and $\tau_\nu = 1$ if B is ramified at ν.

Sketch. In the split case, we have that $B_\nu \cong M_2(F_\nu) \cong \mathrm{End}_{F_\nu}(E_\nu) \supset \mathrm{End}_{\mathcal{O}_{F_\nu}}(\Lambda_\nu)$. We relate $g_\nu \mathcal{O}_\nu g_\nu^{-1}$ and $\mathrm{End}_{\mathcal{O}_{F_\nu}}(\Lambda_\nu)$, which are both maximal orders in $B(F_\nu)$ and are therefore conjugate. Therefore $g_\nu \mathcal{O}_\nu g_\nu^{-1}$ is the endomorphism ring of a $\mathcal{O}_{F_\nu}$-lattice $\mathcal{L} \subset E_\nu$. By definition of Λ_ν this lattice is a proper fractional Λ_ν-ideal, so the two maximal orders are in fact conjugate by an element of $E_\nu^\times$.

In the non-split case, we use the fact that $g_\nu \mathcal{O}_\nu g_\nu^{-1} = \mathcal{O}_\nu$ is determined by the reduced norm. □

In addition, we have a statement for the infinite places:

Proposition 2.4.3. *Let $\nu|\infty$ be an infinite place. If B_ν is non-split, the reduced norm can be calculated on the representation in Proposition 2.4.1 as*

$$\mathrm{Nr}\begin{pmatrix} a & \epsilon_\nu b \\ {}^\sigma b & {}^\sigma a \end{pmatrix} = |a|^2 - \epsilon_\nu |b|^2$$

and is conjugation invariant, so

$$g_\nu \widetilde{\Omega}_\nu g_\nu^{-1} = \left\{ \begin{pmatrix} a & \epsilon_\nu b \\ {}^\sigma b & {}^\sigma a \end{pmatrix} : \frac{1}{4} \le |a|^2 - \epsilon_\nu |b|^2 \le 4 \right\}$$
$$g_\nu \mathcal{O}_\nu g_\nu^{-1} = \left\{ \begin{pmatrix} a & \epsilon_\nu b \\ {}^\sigma b & {}^\sigma a \end{pmatrix} : |a|^2 - \epsilon_\nu |b|^2 \le 1 \right\}$$
$$g_\nu \mathcal{O}_\nu^\times g_\nu^{-1} = \left\{ \begin{pmatrix} a & \epsilon_\nu b \\ {}^\sigma b & {}^\sigma a \end{pmatrix} : |a|^2 - \epsilon_\nu |b|^2 = 1 \right\}$$

Note that $\epsilon_\nu < 0$ since B_ν is non-split, so these are indeed compact. Now suppose that B_ν is split. Then

$$g_\nu \widetilde{\Omega}_\nu g_\nu^{-1} = \left\{ \begin{pmatrix} a & b f_\nu \\ {}^\sigma b / f_\nu & {}^\sigma a \end{pmatrix} : |a| + |b| \le 2, |a| - |b| \ge 1/2 \right\}$$
$$g_\nu \mathcal{O}_\nu g_\nu^{-1} = \left\{ \begin{pmatrix} a & b f_\nu \\ {}^\sigma b / f_\nu & {}^\sigma a \end{pmatrix} : |a| + |b| \le 1 \right\}$$
$$g_\nu \mathcal{O}_\nu^\times g_\nu^{-1} = \left\{ \begin{pmatrix} a & b f_\nu \\ {}^\sigma b / f_\nu & {}^\sigma a \end{pmatrix} : |a| + |b| = 1 \right\}$$

Finally, the critical result of Section 5 of [Kha17] is the following: for an element of the projective unit group, we can find an integral representative which is 'minimal' in some sense. To define this, we need a metric that is compatible with the Cartan decomposition. Such a metric comes from the Bruhat-Tits building.

Definition 2.4.4. *For a finite place ν of F where G splits, let $\mathscr{B}_\nu$ be the Bruhat-Tits building of $G(F_\nu)$. If G is ramified at $\nu \nmid \infty$ let $\mathscr{B}_\nu$ be the connected graph with two vertices[4] corresponding to $G(F_\nu)/\mathcal{K}_\nu$. Denote by d the geodesic distance function on the graph $\mathscr{B}_\nu$ normalised so that the length of each edge is 1.*

If $\nu|\infty$ splits G, set $\mathscr{B}_\nu = G(F_\nu)/N_{G(F_\nu)}(\mathcal{K}_\nu)$. This is the upper half-plane which we equip with the standard hyperbolic distance function d. If B is ramified at $\nu|\infty$ let $\mathscr{B}_\nu$ be a single point with the trivial metric.

For each place ν let x_0 be the point in $\mathscr{B}_\nu$ stabilised by $\mathcal{K}_\nu$. Let q_ν be the size of the residue field k_ν of F at finite places ν, and set $q_\nu = e$ for $\nu|\infty$. Define a continuous function $\mathfrak{d}_\nu : G(F_\nu) \to \mathbb{R}_{>0}$ by

$$\mathfrak{d}_\nu(x_\nu) := q_\nu^{d(x_0, x_\nu \cdot x_0)}.$$

[4] These vertices correspond to whether the reduced norm of any lift to $B_\nu^\times$ is a square or not in $F_\nu^\times$.

Define the continuous function $\mathfrak{d}_f : G(\mathbb{A}_{F,f}) \to \mathbb{N}$ *by*

$$\mathfrak{d}\left((x_\nu)_{\nu\nmid\infty}\right) = \prod_{\nu\nmid\infty} \mathfrak{d}_\nu(x_\nu).$$

The next Proposition (see Proposition 5.21 of [Kha17]) records how we lift optimally. Here, as will be discussed more in the Section 3, for a fixed regular element[5] $a \in A_\nu$,

$$\mathcal{O}_\nu^{(-n,n)} = \bigcap_{-n\le i\le n} a^i \mathcal{O}_\nu a^{-i}$$
$$\mathcal{K}_\nu^{(-n,n)} = \bigcap_{-n\le i\le n} a^i \mathcal{K}_\nu a^{-i}.$$

Recall that $A = \prod_{\nu\in S} A_\nu$ is a fixed product of maximal split tori of G at the S-adic places (where G is assumed to be split).

Proposition 2.4.5. *For a fixed place* ν *of* F *and* $x_\nu \in G(F_\nu)$*, any* $h \in \Omega_\nu x_\nu \Omega_\nu \subset G(F_\nu)$ *has a lift* $r \in \mathcal{O}_\nu \subset B(F_\nu)$ *satisfying* $h = F_\nu^\times r$ *with*

$$v_\nu\left(\mathrm{Nrd}(r)\right) = d(x_0, x_\nu \cdot x_0) \text{ if } \nu \nmid \infty$$
$$2^{-8} \le \left|\mathrm{Nrd}(r)\right|_\nu \mathfrak{d}_\nu(x_\nu) \le 1 \text{ if } \nu|\infty.$$

Furthermore, for $\nu \nmid \infty$ *this valuation is minimal for fixed* h*. Additionally, if* $x_\nu \in A_\nu$ *and* $h \in \mathcal{K}_\nu^{(-n,n)} x_\nu \mathcal{K}_\nu^{(-n,n)}$ *then we can ensure that* $r \in \mathcal{O}_\nu^{(-n,n)}$*.*

Sketch. For a finite place ν of F for which B splits, the quantity $\mathfrak{d}_\nu(x_\nu)$ determines the ratio of $|\cdot|_\nu$ for the two entries of the diagonal matrix in the Cartan decomposition of x_ν. Scaling so that both of these entries are integral, we obtain the result. □

2.4.2 The Unitary Case

Now, we wish to emulate the results of the previous section for the unitary groups. We have an outer form G of $\mathrm{GL}_{r,F}$ with a maximal torus $T \le G$ defined over F such that T_M is split. Furthermore, M contains a quadratic CM extension K/F over which G splits (but T does not). From such data, we would like to get an explicit representation of $G(F)$ as a subgroup of $\mathrm{GL}_r(M)$. Since we do not have an 'accidental' isomorphism, as in the quaternion algebra case, we cannot hope for this to be as clean as the representation in that case. For that reason, we assume in this section that M/F is in fact a Galois extension.

[5]Regular means that $\alpha(a) \ne 1$ for any $\alpha \in \Phi$, the set of roots of G.

It will be useful to fix an outer automorphism in $\mathrm{Aut}(\mathrm{GL}_r(M))$. The automorphism group is well-known. In the case that $r \geq 3$, the Dynkin diagram A_{r-1} has a non-trivial automorphism, which can be defined over $\mathbb{Z}$, given by

$$\varphi : A \mapsto A^{-T}$$

Picking this representation is a convenient splitting of the exact sequence

$$1 \to \mathrm{PGL}_r(M) \to \mathrm{Aut}(GL_r(M)) \to \mathrm{Out}(GL_r(M)) \to 1$$

where the group $\mathrm{Out}(\mathrm{GL}_r(M))$ is the group of outer automorphisms (isomorphic to the group of automorphisms of the Dynkin diagram). This gives an isomorphism

$$\begin{aligned} \mathrm{PGL}_r(M) \rtimes \mathbb{Z}/2 &\longrightarrow \mathrm{Aut}(\mathrm{GL}_r(M)) \\ (\theta, \epsilon) &\longmapsto \left(A \mapsto \theta\varphi^{\epsilon}(A)\theta^{-1}\right). \end{aligned}$$

Proposition 2.4.6. *Let $r \geq 3$. The outer forms of $\mathrm{GL}_{r,F}$ which split over K are in bijection with matrices $\Theta \in \mathrm{PGL}_2(K)$ satisfying*

$$\overline{\Theta} = \Theta^T = \varphi(\Theta)^{-1}.$$

Furthermore, all injective homomorphisms $G_L \to \mathrm{GL}_{r,L}$ are conjugate for any CM field L/K. Thus an injection $G_M \to \mathrm{GL}_{r,M}$ is determined by Θ as above, and $\vartheta \in \mathrm{PGL}_r(M)$.

Proof. The forms of $\mathrm{GL}_{r,F}$ which split over M are in bijection with the elements of

$$\mathrm{H}^1(\mathrm{Gal}(M/F), \mathrm{Aut}(\mathrm{GL}_r(M))).$$

In fact, the subgroups $G(F) \subset \mathrm{GL}_r(M)$ coming from such forms are in bijection with the cocycles $Z^1(\mathrm{Gal}(M/F), \mathrm{Aut}(\mathrm{GL}_r(M)))$ and two such subgroups are isomorphic if and only if these cocycles are related via a coboundary.

The Galois group $\mathrm{Gal}(M/F)$ acts on $\mathrm{Aut}(\mathrm{GL}_r(M))$ in the following way

$$\sigma \cdot (\theta, \epsilon) = ({}^{\sigma}\theta, \epsilon)$$

that is, it acts purely on the inner automorphism by the natural action. Suppose that

$$c : \mathrm{Gal}\,(M/F) \to \mathrm{Aut}(\mathrm{GL}_r(M)), c(\sigma) = (\theta_\sigma, \epsilon_\sigma)$$

with $\theta_\sigma \in \mathrm{PGL}_r(M)$ (often we consider θ as a matrix in $\mathrm{GL}_r(M)$ for convenience), and $\epsilon_\sigma \in \mathbb{Z}/2$. Then the cocycle condition for c becomes

$$\theta_{\sigma\rho} = \theta_\sigma \varphi^{\epsilon_\sigma}({}^{\sigma}\theta_\rho), \epsilon_{\sigma\rho} = \epsilon_\sigma + \epsilon_\rho.$$

Given that G splits over K, we must have that this cocycle is in the kernel of the restriction map,

$$\mathrm{H}^1(\mathrm{Gal}\,(M/F)\,, \mathrm{Aut}(\mathrm{GL}_r(M))) \to \mathrm{H}^1(\mathrm{Gal}\,(M/K)\,, \mathrm{Aut}(\mathrm{GL}_r(M))).$$

Therefore, there exists $\vartheta \in \mathrm{PGL}_r(M), \epsilon_0 \in \mathbb{Z}/2$ such that for all $\sigma \in \mathrm{Gal}(M/K)$,

$$\begin{aligned}(\theta_\sigma, \epsilon_\sigma) &= (\vartheta, \epsilon)^{-1}({}^\sigma\vartheta, \epsilon) \\ &= (\varphi^\epsilon(\vartheta^{-1}), \epsilon)({}^\sigma\vartheta, \epsilon) \\ &= (\varphi^\epsilon(\vartheta^{-1}\,{}^\sigma\vartheta), 0)\end{aligned}$$

After possibly replacing ϑ with $\varphi(\vartheta)$, this means that $\theta_\sigma = \vartheta^{-1}\,{}^\sigma\vartheta$ and $\epsilon_\sigma = 0$. In fact, this shows that the map

$$B^1(\mathrm{Gal}\,(M/F)\,, PGL_r(M)) \to B^1(\mathrm{Gal}\,(M/F)\,, \mathrm{Aut}(\mathrm{GL}_r(M)))$$

is surjective, proving that if two isomorphic such forms of $\mathrm{GL}_r(F)$ inside $\mathrm{GL}_r(M)$ must be conjugate.

Denote complex conjugation on M by $\tau \in \mathrm{Gal}(M/F)$. Since G is not an inner form of $\mathrm{GL}_{r,F}$, we must have $\epsilon_\tau = 1$. The condition $\tau^2 = 1$ implies that

$$\theta_\tau \theta_\tau^{-T} = 1. \tag{2.3}$$

Finally, the relation $\sigma\tau = \tau\sigma$, gives the equation

$$\begin{aligned}\vartheta^{-1}\,{}^\sigma\vartheta\,{}^\sigma\theta_\tau = \theta_\tau\varphi({}^\tau\vartheta^{-1}\,{}^{\sigma\tau}\vartheta), \forall\sigma \in \mathrm{Gal}(M/K) \\ \implies \vartheta\theta_\tau\varphi({}^\tau\vartheta^{-1}) \in \mathrm{GL}_r(K).\end{aligned}$$

If we set $\Theta = \vartheta\theta_\tau\varphi({}^\tau\vartheta^{-1})$, then (2.3) reduces simply to

$${}^\tau\Theta = \Theta^T. \tag{2.4}$$

Therefore, a choice of an outer form is simply given by a choice of $\vartheta \in \mathrm{PGL}_r(M)$ and $\Theta \in \mathrm{PGL}_r(K)$ satisfying (2.4). A simple computation shows that altering this pair by a coboundary simply multiplies ϑ by the corresponding element in $\mathrm{PGL}_r(M)$ for the coboundary. Therefore, Θ uniquely determines the unitary group. □

Proposition 2.4.7. *Given a pair $(\Theta, \vartheta) \in \mathrm{PGL}_r(K) \times \mathrm{PGL}_r(M)$ defining an embedding $G_M \to \mathrm{GL}_{r,M}$, the F-rational points of G inside $\mathrm{GL}_{r,M}$ are the fixed points of the automorphisms*

$$\begin{aligned}A &\mapsto \mathrm{Ad}(\vartheta^{-1}\,{}^\sigma\vartheta)\,{}^\sigma A, \forall\sigma \in \mathrm{Gal}(M/K) \\ A &\mapsto \mathrm{Ad}(\vartheta^{-1}\Theta\varphi({}^\tau\vartheta))\varphi({}^\tau A).\end{aligned}$$

In fact, these automorphisms define the Galois action of $\mathrm{Gal}(M/F)$ *on* $\mathrm{GL}_r(M)$ *coming from the base change isomorphism*

$$G(M) \xrightarrow{\sim} \mathrm{GL}_r(M).$$

Proof. This comes from the fact that given a cocycle

$$c \in \mathrm{Z}^1(\mathrm{Gal}(M/F), \mathrm{Aut}(\mathrm{GL}_r(M)))$$

there is a twisted action of $\mathrm{Gal}(M/F)$ by

$$\rho * A = c(\rho)({}^{\rho}A).$$

The proof of Proposition 2.4.6 shows that the cocycle corresponding to (Θ, θ) is

$$c(\sigma) = \mathrm{Ad}(\vartheta^{-1}\, {}^{\sigma}\vartheta), \forall \sigma \in \mathrm{Gal}\,(M/K)\,,$$

and

$$c(\tau) = \mathrm{Ad}(\vartheta^{-1}\Theta\varphi({}^{\tau}\vartheta)) \circ \varphi.$$

□

Given an M-split torus $T \leq G$, we can choose $(\Theta, \vartheta) \in \mathrm{PGL}_r(K) \times \mathrm{PGL}_r(M)$ such that $T(M)$ is the diagonal torus of $\mathrm{GL}_r(M)$. Define $S \leq \mathrm{PGL}_{r,M}$ to be the diagonal torus, and $\widetilde{S} \leq \mathrm{GL}_{r,M}$ its pre-image (also called the diagonal torus).

Proposition 2.4.8. *Suppose* (Θ, ϑ) *are chosen such that the diagonal matrices* $\widetilde{S} \leq \mathrm{GL}_{r,M}$ *are the base change* T_M *of an* F*-rational torus* $T \leq G$ *with splitting field* M*. Up to conjugacy by* $N_{\mathrm{GL}_r(M)}(\widetilde{S})$*, such a choice is equivalent to a choice of* (f, ζ) *where*

- f *is a (representative of a) conjugacy class of homomorphisms* $\mathrm{Gal}(M/K) \to S_r$*, where* S_r *is the symmetric group, such that when* $S_r \hookrightarrow \mathrm{GL}_r(\mathbb{Z})$ *(via permutation matrices) this representation becomes conjugate to the regular representation of* $\mathrm{Gal}(M/K)$*; and,*
- $\zeta \in S(M^+)$ *satisfies*

$$f(\sigma)\, {}^{\sigma}\zeta f(\sigma)^{-1} = \zeta, \forall \sigma \in \mathrm{Gal}(M/K)$$

In this case, the F*-rational points are the fixed points of*

$$\begin{aligned} A &\mapsto \mathrm{Ad}(f(\sigma))^{\sigma}A, \forall \sigma \in \mathrm{Gal}(M/K) \\ A &\mapsto \mathrm{Ad}(\zeta)\varphi({}^{\tau}A). \end{aligned}$$

Proof. Since $\widetilde{S} = T_M$ arises via base change from an F-rational torus, the twisted Galois action must preserve $\widetilde{S}$. Since φ preserves the diagonal torus (hence why we chose this particular outer automorphism), this amounts to

$$\vartheta^{-1}\,{}^{\sigma}\vartheta, \vartheta^{-1}\Theta\varphi({}^{\tau}\vartheta) \in N_{\mathrm{PGL}_r(M)}(S), \forall \sigma \in \mathrm{Gal}(M/K).$$

Let $N := N_{\mathrm{PGL}_r(M)}(S) = S(M) \ltimes S_r$ be the normaliser of the diagonal matrices, where we choose a splitting of S_r (the symmetric group on r letters) given by the permutation matrices. Then the cocycle c lies in the kernel of the map

$$\mathrm{H}^1(\mathrm{Gal}(M/K), N) \to \mathrm{H}^1(\mathrm{Gal}(M/K), \mathrm{PGL}_r(M))$$

Using Hilbert 90 and the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{GL}_1(M) & \longrightarrow & \mathrm{GL}_r(M) & \longrightarrow & \mathrm{PGL}_r(M) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathrm{GL}_1(M) & \longrightarrow & \widetilde{N} & \longrightarrow & N & \longrightarrow & 1 \end{array}$$

we find that this kernel is the image of $\mathrm{H}^1(\mathrm{Gal}(M/K), \widetilde{N})$. Using the exact sequence

$$1 \to S(M) \to N \to S_r \to 1,$$

and Hilbert 90 again, we see that

$$\mathrm{H}^1(\mathrm{Gal}(M/K), \widetilde{N}) = \mathrm{H}^1(\mathrm{Gal}(M/K), N) = \mathrm{Hom}(\mathrm{Gal}(M/K), S_r)/\mathrm{conj}$$

This means that the data of ϑ is equivalent to choosing (up to conjugacy) some $f \in \mathrm{Hom}(\mathrm{Gal}(M/K), S_r)$, and $\xi \in S(M)$. Then

$$\vartheta^{-1}\,{}^{\sigma}\vartheta = \xi^{-1} f(\sigma)^{\sigma}\xi, \forall \sigma \in \mathrm{Gal}(M/K).$$

By replacing ϑ with $\vartheta\xi^{-1}$, we can assume that $\xi = 1$.

If we let $X = \vartheta^{-1}\Theta\varphi({}^{\tau}\vartheta) \in N_{\mathrm{PGL}_r(M)}(S)$, then the conditions on Θ are equivalent to

$$f(\sigma)\,{}^{\sigma}X\varphi(f(\sigma))^{-1} = X, \forall \sigma \in \mathrm{Gal}(M/K)$$
$${}^{\tau}X = \varphi(X)^{-1}.$$

Suppose now that we write $X = \xi\pi_0$ where $\xi \in S(M)$ and π_0 is an elementary matrix (equivalently an element of S_r). The conditions above are precisely that

$\pi_0 = \pi_0^T$, i.e. the corresponding permutation has order 2, and

$$f(\sigma)^\sigma \xi f(\sigma)^{-1} = \xi$$
$$\pi_0 f(\sigma) = f(\sigma)\pi_0$$
$${}^\tau\xi_i = \xi_{\pi_0(i)}.$$

Finally, we need to verify that the representation $\widehat{T}$ corresponding to the fixed points constructed here is isomorphic to the character lattice of the torus we started with (currently we just have some torus contained in G which splits over M, not necessarily the torus we started with). There is a natural basis for the character lattice of the diagonal torus $\widetilde{S}$, given by

$$\psi_i : A \mapsto A_i.$$

Using the Galois action on points given above, the action on the character lattice is therefore

$$\sigma \cdot \psi_i = \psi_{f(\sigma)(i)}$$
$$\tau \cdot \psi_i = -\psi_{\pi_0(i)}.$$

This must be isomorphic as a $\mathbb{Z}[\mathrm{Gal}(M/F)]$-module to $\widehat{T}$. This implies that $\pi_0 = \mathrm{id}$, and that the representation

$$f : \mathrm{Gal}(M/K) \to \mathrm{GL}_r(\mathbb{Z})$$

is conjugate to the regular representation of $\mathrm{Gal}(M/K)$ on itself. □

This is simplified in the cyclic case by the following group theoretic lemma.

Lemma 2.4.9. *Suppose that*

$$\phi_1, \phi_2 : H \to S_r$$

are two injective homomorphisms from a cyclic group H into the symmetric group such that when $S_r \hookrightarrow \mathrm{GL}_r(\mathbb{Z})$ via the elementary matrices, they become conjugate. Then in fact $\phi_1(H), \phi_2(H)$ are conjugate in S_r.

It is not clear to us whether this result may hold more generally for abelian or non-abelian subgroups. It is certainly true that there are isomorphic non-conjugate subgroups of S_r, however the proof below will show that we require such subgroups where there is an isomorphism preserving the cycle type. Maybe it is true that cycle-type preserving isomorphisms between subgroups of S_r can be realised via conjugation.

Proof. Given an element of $S_r \subset \mathrm{GL}_r(\mathbb{Z})$, we can determine its cycle type with just $\mathrm{GL}_r(\mathbb{Z})$-invariant calculations (using the fact that the number of cycles is equal to the dimension of the fixed space, and the cycle type is determined by the number of cycles of all powers). Therefore, $\phi_1(h)$ is conjugate to $\phi_2(h)$ for all $h \in H$. Applying this to a generator of H implies the result. □

From now on, assume that the Galois group $\mathrm{Gal}(M/K)$ is cyclic (or more generally, satisfies Lemma 2.4.9). This means that we may identify the Galois group $\mathrm{Gal}(M/K)$, the set of field embeddings $\mathrm{Hom}_K(M, \overline{K})$, and the indices of the matrix representation in Proposition 2.4.8. We let $\sigma_i \leftrightarrow f_i \leftrightarrow i$ be this correspondence.

Corollary 2.4.10. *We can embed $M \hookrightarrow K^r$ in such a way that $T(K) \cong M^\times$ acts by multiplication. Furthermore the K-rational points $G(K) \cong \mathrm{GL}_r(K)$ are then naturally identified with the subset of invertible elements of $M^r = \mathrm{End}_K(M)$ where the i-th component acts on M via $y \cdot m = y^{\sigma_i} m$.*

Explicitly, the points $G(K)$ are the invertible matrices of the form

$$\begin{pmatrix} x_1 & \dots & x_r \\ {}^{\sigma_2}x_{\sigma_2^{-1}(1)} & \dots & {}^{\sigma_2}x_{\sigma_2^{-1}(r)} \\ \vdots & & \vdots \\ x_{\sigma_r^{-1}(1)} & \dots & {}^{\sigma_r}x_1 \end{pmatrix}.$$

The embedding $m \mapsto (m, {}^{\sigma_2}m, \dots, {}^{\sigma_r}m)^T$ then realises the previous statement.

Furthermore, there exists $\epsilon \in M^{+,\times}$ such that the F-rational points are then given by matrices of the above form that additionally satisfy

$$\sum_{i=1}^{r} {}^{\sigma_i}\epsilon x_i \, {}^{\tau}x_i = 1$$

$$\sum_{i=1}^{r} {}^{\sigma_i}\epsilon x_i \, {}^{\tau\sigma_j}(x_{\sigma_j^{-1}(i)}) = 0, \forall j \neq 1.$$

Proof. Firstly, the linear independence of field automorphisms implies that $\mathrm{End}_K(M) \cong M^r$ with the given action.

The condition $f(\sigma)^\sigma \zeta f(\sigma)^{-1} = \zeta$ implies that $\zeta_{\sigma(i)} = {}^\sigma\zeta_i$, and so by scaling we can assume that for some $\epsilon \in M^{+,\times}$, $\zeta_{\sigma(i)} = \epsilon / {}^\sigma\epsilon$. The K-points of G are then given by matrices $A \in \mathrm{GL}_r(M)$ such that

$$A_{\sigma(i),\sigma(j)} = {}^\sigma A_{i,j}.$$

This proves the statements about $G(K)$. The twisted action of complex conjugation on $G(M)$ is given by

$$A \mapsto \zeta\, {}^{\tau}A^{-T}\zeta^{-1}$$

The conditions for $G(F)$ follow immediately from this. $\square$

Corollary 2.4.10 gives an embedding of $G(K)$ into $\mathrm{End}_K(M)$. This contains a maximal order $\mathrm{End}_{\mathcal{O}_K}(\Lambda)$. We now almost give an explicit description of this maximal order. It will be convenient to consider the matrix representation in the Corollary 2.4.10 via just the first row.

Proposition 2.4.11. *The elements of* $\mathrm{End}_K(\Lambda)$ *in the representation of Corollary 2.4.10 satisfy* $x_i \in \widehat{\Lambda}, \forall i$. *If we choose some finite place* ν *of* F*, then there exists* $\tau_\nu \in M_\nu^\times$ *such that every element of* $g_\nu \mathcal{O}_\nu g_\nu^{-1}$ *has the form*

$$\left(x_1, \frac{\tau_\nu}{{}^{\sigma_2}\tau_\nu}x_2, ..., \frac{\tau_\nu}{{}^{\sigma_r}\tau_\nu}x_r\right), x_i \in \widehat{\Lambda}_\nu \forall i.$$

Proof. We require that for every $m \in \Lambda$,

$$\sum_{i=1}^{r} x_i\, {}^{\sigma_i}m \in \Lambda.$$

Any $l \in \Lambda$ and σ_j gives a K-linear endomorphism of Λ by $m \mapsto l^{\sigma_j}m$. Multiplying on the right by this corresponds to

$$(x_1, ..., x_n) \mapsto ({}^{\sigma_j^{-1}}lx_{\sigma_j^{-1}(1)}, {}^{\sigma_2\sigma_j^{-1}}lx_{\sigma_2\sigma_j^{-1}(1)}, ..., {}^{\sigma_r\sigma_j^{-1}}lx_{\sigma_r\sigma_j^{-1}}).$$

These must all have integral trace, since any element of a maximal order of $\mathrm{End}_K(M)$ is integral. Therefore, we see that

$$\mathrm{Tr}(x_{\sigma_j^{-1}(1)}\, {}^{\sigma_j^{-1}}l) \in \mathcal{O}_K, \forall l \in \Lambda, \forall j.$$

This implies that $x_1, ..., x_r \in \widehat{\Lambda}$. Now, since $g_\nu \mathcal{O}_\nu g_\nu^{-1} = \mathrm{End}_{K_\nu}(\tau_\nu \Lambda_\nu)$ for some τ_ν, we obtain the desired result. $\square$

2.5 Volume, Discriminant and Order of Toral Sets

To a homogeneous toral set we can assign a volume given a choice of a compact neighbourhood of the identity in $\mathbb{P}(\mathbb{A})$. The algebraic group T_i is by assumption anisotropic, and therefore $\mathbb{T}_i(\mathbb{A})$ has a bi-invariant Haar measure giving the subgroup

$\mathbb{T}_i(\mathbb{Q})$ covolume 1, since the quotient is compact. Recall, from Section 2.1, that we are considering an open compact subgroup (therefore a neighbourhood of the identity)

$$\mathcal{K}^\infty = \prod_{\nu\nmid\infty} \mathcal{K}_\nu \subset P(\mathbb{A}_F^\infty)$$

and $\mathcal{K}_{G,\infty} \subset G(F_\infty)$ a maximal compact torus.

Definition 2.5.1. *Let Ω_∞ be a compact $\mathcal{K}_{G,\infty}$-invariant neighbourhood of $\mathbb{P}(\mathbb{R})$ (such as the ones constructed in Section 2.3). Then $\Omega_\mathbb{A} := \mathcal{K}^\infty \times \Omega_\infty \subset \mathbb{P}(\mathbb{A})$ is a compact open neighbourhood of the identity, and $\Omega_\mathbb{A}^2 \subset \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{A})$ is the same. Then,*

- *To the homogeneous toral set $[\mathcal{T}\xi]$, we assign the volume*

$$\mathrm{vol}([\mathcal{T}\xi]) := m_\mathcal{T}(\mathrm{Ad}_\xi\Omega_\mathbb{A} \cap \mathcal{T})^{-1} = [\mathbb{T}(\mathbb{A}) : \mathcal{T}]^{-1}\, m_{\mathbb{T}(\mathbb{A})}\left(\mathrm{Ad}_\xi\Omega_\mathbb{A} \cap \mathcal{T}\right)^{-1}.$$

  Here $m_\mathcal{T}$ is the total volume 1 Haar measure on the image of $\mathcal{T}$ in $\mathbb{P}(\mathbb{Q})\setminus\mathbb{P}(\mathbb{A})$.

- *To the homogeneous toral set $[\mathcal{T}(\xi, s\xi)]$, we assign the same volume*

$$\mathrm{vol}([\mathcal{T}(\xi, s\xi)]) = m_\mathcal{T}(\mathrm{Ad}_\xi\Omega_\mathbb{A} \cap \mathcal{T})^{-1} = [\mathbb{T}(\mathbb{A}) : \mathcal{T}]^{-1}\, m_{\mathbb{T}(\mathbb{A})}\left(\mathrm{Ad}_\xi\Omega_\mathbb{A} \cap \mathcal{T}\right)^{-1}.$$

  Note that in this case, we will in the mixing case generally have $V = 0$ and so we often write g instead of ξ.

The reason for no appearance of s in the second definition is that $\mathrm{Ad}_{s\xi}\Omega_\mathbb{A} \cap \mathcal{T} = \mathrm{Ad}_\xi\Omega_\mathbb{A} \cap \mathcal{T}$. If we assume that $\xi_\infty^{-1}\mathbb{T}(\mathbb{R})\xi_\infty \leq \mathcal{K}_\infty$, then the choice of Ω_∞ becomes irrelevant and we simply obtain

$$\mathrm{vol}([\mathcal{T}\xi]) = m_\mathcal{T}(\mathcal{K}_\mathcal{T})^{-1} = [\mathbb{T}(\mathbb{A}) : \mathcal{T}]^{-1}\, m_{\mathbb{T}(\mathbb{A}_f)}\left(\mathcal{K}_\mathcal{T}\right)^{-1},$$

where $\mathcal{K}_\mathcal{T} = \mathrm{Ad}_\xi\mathcal{K}_f \cap \mathcal{T}$.

To an algebraic torus, $\mathbb{T}$ we can attach a collection of invariants $h_\mathbb{T}, w_\mathbb{T}, D_\mathbb{T}, \rho_\mathbb{T}, R_\mathbb{T}, r_\mathbb{T}, s_\mathbb{T}$ as in the work of Ono and Shyr (see [Ono61] and [Shy77]), analogously to the case of the multiplicative group of a number field. Using these standard definitions, we note that for a $\mathcal{K}_\infty$-invariant homogeneous toral set $[\mathcal{T}\xi]$, the volume is given by

$$\mathrm{vol}([\mathcal{T}\xi]) = [\mathbb{T}(\mathbb{A}) : \mathcal{T}]^{-1}\left[\mathcal{K}_f^{max} : \mathrm{Ad}_{\xi_f}\mathcal{K}_f \cap \mathcal{T}\right]\frac{h_\mathbb{T}}{w_\mathbb{T}},$$

where $\mathcal{K}_f^{max} \subset \mathbb{T}(\mathbb{A}_f)$ is any choice of maximal compact subset containing $\mathrm{Ad}_{\xi_f}\mathcal{K}_f \cap \mathcal{T}$. We will give a more detailed computation of the volume in Section 4.

We now give a discriminant of a $\mathcal{K}_\infty$-invariant homogeneous toral set following [Ein+06]. Notice that this depends only on $\mathbb{T}(\mathbb{A})$ and not on the chosen subgroup $\mathcal{T}$.

Definition 2.5.2. *Fix a* $\mathbb{Z}$*-Lie algebra* $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ *such that* $B(\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}) \subset \mathbb{Z}$. *For each prime* p *this induces a* $\mathbb{Z}_p$*-Lie algebra* $\mathfrak{g}_{\mathbb{Z}_p} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset \mathfrak{g}_{\mathbb{Q}_p}$. *Given a maximal torus* $T_p \subset \mathbb{G}(\mathbb{Q}_p)$*, we can associate an element* $x_{T_p} \subset \left(\bigwedge^r \mathfrak{g}_{\mathbb{Q}_p}\right)^{\otimes 2}$ *in the standard way by taking a* $\mathbb{Q}_p$*-basis* $f_1, ..., f_r$ *for* $\mathfrak{t}_p$ *(the Lie algebra of* T_p*) and setting*

$$x_{T_p} = \left[(f_1 \wedge ... \wedge f_r)^{\otimes 2} \left(\det B(f_i, f_j)\right)^{-1}\right].$$

To a homogeneous toral set $[\mathcal{T}l] \subset [\mathbb{G}(\mathbb{A})]$ *that is invariant under* $\mathcal{K}_\infty$*, we assign the discriminant*

$$\operatorname{disc}\left([\mathcal{T}l]\right) = \prod_p \operatorname{disc}_p\left([\mathcal{T}l]\right),$$

where

$$\operatorname{disc}_p([\mathcal{T}l]) = |x_{l_p^{-1}\mathfrak{t}_p l_p}|_{\mathfrak{g}_{\mathbb{Z}_p}}$$

and the norm, $|\cdot|_{\mathfrak{g}_{\mathbb{Z}_p}}$ *is the standard norm with unit ball coming from* $\mathfrak{g}_{\mathbb{Z}_p}$.

The discriminant is a measure of the denominators required to describe the semi-simple part of a homogeneous toral set (even if we are considering a case where $V \neq 0$, we only use the projection of the toral set to $\mathfrak{G}(\mathbb{A})$ to define the discriminant). We relate this to the discriminant of the order attached to $[\mathcal{T}l]$ in the next result.

Proposition 2.5.3. *In both the quaternionic and unitary cases, up to a multiplicative constant,* c_F*, bounded uniformly for fixed* F*, the discriminant can be computed from the order* Λ*. More precisely,*

$$\operatorname{disc}([\mathcal{T}g]) = \begin{cases} c_F \operatorname{disc}(\Lambda), & \textit{in the quaternionic case} \\ c_F \operatorname{disc}(\Lambda)/\operatorname{disc}(\Lambda \cap M^+), & \textit{in the unitary case.} \end{cases}$$

where all discriminants are absolute (i.e. over $\mathbb{Z}$*).*

Proof. Fix a $\mathbb{Z}$-basis $\{\alpha_1, ..., \alpha_d\}$ of $\mathcal{O}_F$.

In the representations given above, the torus $\mathbb{T}$ corresponds to the diagonal torus. In the quaternionic case, the Lie algebra of the determinant 1 elements of the diagonal torus becomes

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \alpha \in E^{\operatorname{tr}_{E/F}=0} \right\}$$

Therefore if x is an $\mathcal{O}_{F_\nu}$-generator for $\mathfrak{t}_\nu \cap g_\nu \mathcal{O}_\nu g_\nu^{-1} = \Lambda_\nu^{\operatorname{tr}=0}$, we get that $\{\alpha_i x\}_i$ a $\mathbb{Z}$-basis, and so

$$\operatorname{disc}_p([\mathcal{T}g]) = \left| \frac{(x\alpha_1 \otimes ... \otimes x\alpha_d)^{\otimes 2}}{\det \operatorname{Tr}(x^2 \alpha_i \alpha_j)} \right|_p = \left|\det \operatorname{Tr}(x^2\alpha_i\alpha_j)\right|_p = \frac{\operatorname{disc}(\mathcal{O}_{F_\nu} \oplus \mathcal{O}_{F_\nu} x)}{\operatorname{disc}(\mathcal{O}_{F_\nu})}.$$

When $p \neq 2$, the numerator is equal to the discriminant of Λ, and when $p = 2$ it differs by at most $2^{[F:\mathbb{Q}]}$ and so the two global quantities are equal up to possibly this constant.

In the unitary case, we similarly get the discriminant as

$$\mathrm{disc}_p([\mathcal{T}g]) = \mathrm{disc}\left(\Lambda_p^{\mathrm{Tr}_{M/M^+}=0=\mathrm{Tr}_{M/K}}\right).$$

Up to a uniformly bounded constant (depending only on $[F:\mathbb{Q}]$ and r), this gives the global discriminant as

$$\mathrm{disc}([\mathcal{T}l]) \sim \frac{\mathrm{disc}(\Lambda)}{\mathrm{disc}(\mathcal{O}_F)\mathrm{disc}(\Lambda \cap M^+)}$$

□

Henceforth, we will use $\mathrm{disc}([\mathcal{T}g])$ and $\mathrm{disc}(\Lambda)$ interchangeably in the quaternion algebra case. In the case that there is also a unipotent part (i.e. the Kuga-Sato case), we also need a measure of the denominators required in the unipotent part.

Definition 2.5.4. *Let $[\mathcal{T}(l,x)] \subset [\mathbb{G}(\mathbb{A}) \ltimes \mathbb{V}(\mathbb{A})]$ be a homogeneous toral set. Then*

$$\mathrm{ord}\left([\mathcal{T}(l,x)]\right) := \prod_p \mathrm{ord}_p\left([\mathcal{T}(l,x)]\right), \mathrm{ord}_p\left([\mathcal{T}(l,x)]\right) := \mathrm{ord}_{\mathbb{V}(\mathbb{Q}_p)/\mathbb{V}(\mathbb{Z}_p)}(l_p^{-1}x_p).$$

2.6 Main Results

We now give the results that we prove in this thesis. First, we discuss the Kuga-Sato results.

Definition 2.6.1. *A sequence of $\mathcal{K}_\infty$-invariant homogeneous toral sets $\{[\mathcal{T}_i(l_i,x_i)]\}_i$ of $\mathbb{P}(\mathbb{A})$ is called strict if $\mathrm{disc}(\mathcal{T}_i l_i) \to \infty$ as $i \to \infty$ and for every non-zero linear map $\phi : \mathbb{V} \to \mathbb{W}$ of $\mathbb{G}$-representations,*

$$\left\{\phi(l_i^{-1}x_i)\right\}_i \in \mathbb{W}(\mathbb{A})$$

escapes all compact sets in $\mathbb{W}(\mathbb{A})$.

The following conjecture states that strictness is sufficient to bootstrap single equidistribution with $V = 0$ to setting with $V \neq 0$. Since we consider only an ergodic approach to this question, we additionally suppose that there are splitting conditions and boundedness conditions at a fixed set of places S which allow the ergodic method to apply - it is possible the conjecture holds without these conditions. It does not appear to have been stated previously in the literature, however is a clear expectation after the work [Kha19b]. We use subgroups $\mathbb{G}(\mathbb{A})^+ := \mathrm{im}(\mathbb{G}^{sc}(\mathbb{A}) \to \mathbb{G}(\mathbb{A}))$, and $\mathbb{P}(\mathbb{A})^+ := \mathbb{G}(\mathbb{A})^+ \ltimes \mathbb{V}(\mathbb{A})$, which will be discussed more in Section 3.

Conjecture 2.6.2. *Let $\mathbb{G}$ be a semi-simple group over $\mathbb{Q}$ and $\mathbb{V}$ be a rational linear representation of $\mathbb{G}$ such that $\mathbb{V}$ does not contain the trivial representation. Let $[\mathcal{T}_i(l_i, x_i)] \subset [\mathbb{P}(\mathbb{A})]$ be a strict sequence of $\mathcal{K}_\infty$-invariant homogeneous toral sets, with associated periodic probability measures μ_i (the pushforward of the Haar measure on $\mathcal{T}_i$ normalised to have mass 1).*

Assume that

1. *the tori $\mathbb{T}_i$ split at the places in S, and $|S|\mathrm{rank}(\mathbb{G}) > 1$,*
2. $\forall p \in S, \mathrm{disc}_p([\mathcal{T}_i(l_i, x_i)]) \ll 1$,
3. $\forall p \in S, \mathrm{ord}_p([\mathcal{T}_i(l_i, x_i)]) \ll 1$,
4. *Any weak-* limit of the projections $\pi_{\mathbb{G}}\mu_i$ is $\mathbb{G}(\mathbb{A})^+$-invariant.*

Then the sequence $\{\mu_i\}_i$ is tight and if $\mu_i \to \mu$ on $[\mathbb{P}(\mathbb{A})]$ then μ is right invariant under $\mathbb{P}(\mathbb{A})^+$.

Proposition 2.6.3. *Given $\mathbb{G}, \mathbb{V}$ as above, if the conjecture holds for all irreducible G-representations $\mathbb{W}$ which have a non-trivial constituent in $\mathbb{V}$ (and holds for the zero representation), then it holds for $\mathbb{V}$.*

The following Theorem is not strictly proven in this thesis, however we provide all the necessary components and simply sketch the proof for the sake of time and length. See Section 8.1.

Theorem 2.6.4. *The conjecture holds for $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_{2,F}$ for any totally real field F, and the representation $\mathbb{G}_a^2$.*

Now we discuss the main result of this thesis. This proves mixing of CM orbits in the quaternion setting over totally real fields under a number of assumptions. See Section 8.2 for the proof. First, we recap the definitions and make some extra assumptions.

We have a quaternion algebra B defined over a totally real field, F, (with $[F : \mathbb{Q}] = d$) from which we defined the algebraic group $G = PB^\times$ of projective units in the quaternion algebra, and we take a finite set, S, of finite places of $\mathbb{Q}$ where F splits completely, B splits at every place of F lying above S, and $|S|d > 1$. We have fixed a maximal compact torus $\mathcal{K}_\infty < G(F_\infty)$. We have a sequence of F-algebra embeddings $E_i \to B$ of quadratic CM extensions E_i/F into B, which induce maximal rank anisotropic tori $T_i < G$. For each torus T_i, we have $g_i \in G(\mathbb{A}_F)$, $s_i \in T_i(\mathbb{A}_F)$, and a finite index subgroup $\mathcal{T}_i < T_i(\mathbb{A}_F)$. To these, we associate the homogeneous joint toral set

$$[\mathcal{T}_i(g_i, s_i g_i)] \subset (G(F) \setminus G(\mathbb{A}_F))^2 .$$

We assume these are $\mathcal{K}_\infty$-invariant, in the sense that $g_i^{-1} T_i(F_\infty) g_i = \mathcal{K}_\infty$.

We can associate to these toral sets an order $\Lambda_i \subset E_i$ with discriminant D_i and conductor $\mathfrak{f}_i$, as in 2.3.1. This order depends of T_i and g_i but not on $\mathcal{T}_i$ or the twist s_i. Recall that the conductor satisfies $D_{\Lambda_i/\mathcal{O}_F} = D_{\mathcal{O}_{E_i}/\mathcal{O}_F} \mathfrak{f}_i^2$, where these are the relative discriminants (which are ideals of $\mathcal{O}_F$). Furthermore, the absolute discriminant is given by

$$D_i = N(D_{\Lambda_i/\mathcal{O}_F}) D_F^2 = N \mathfrak{f}_i^2 D_{E_i}.$$

We assume the following conditions on the subgroup $\mathcal{T}_i$:

1. $\mathcal{T}_i = T_i(F_\infty) \prod_\nu \mathcal{T}_{i,\nu}$ splits as a product over the places of F. In addition, assume that $\mathcal{T}_{i,\nu} = T_i(F_\nu)$ at all places ν where B is ramified.

2. $\mathcal{T}_i$ corresponds to a subgroup of the class group

$$T_i(F) \setminus T_i(\mathbb{A}_F^\infty) / (T_i(\mathbb{A}_F^\infty) \cap g_{i,f} \mathcal{K}_f g_{i,f}^{-1})$$

(which is related to the class group of the order Λ_i constructed in 2.3.1). In particular $T_i(F) \left(T_i(\mathbb{A}_F^\infty) \cap g_{i,f} \mathcal{K}_f g_{i,f}^{-1} \right) < \mathcal{T}_i$.

3. $\mathcal{T}_i$ contains the intersection $T_i(\mathbb{A}_F) \cap G(\mathbb{A}_F)^+ = \mathrm{im}(B^{(1)}(\mathbb{A}_F) \to G(\mathbb{A}_F))$, where $B^{(1)}$ is the algebraic group of norm 1 elements of B.

4. $\mathcal{T}_i$ is preserved by the Galois action of $\mathrm{Gal}(E/F)$.

With the exception of condition (3), these are made for convenience and not expected to be of particular importance to the result of Theorem 2.6.5. The third condition is important in applying the measure theoretic classifications of Section 3. To such a homogeneous toral set we associated, in Definition 2.5.2, a discriminant, $\mathrm{disc}([\mathcal{T}_i(g_i, s_i g_i)]) = \prod_p \mathrm{disc}([\mathcal{T}_{i,p}(g_{i,p}, s_{i,p} g_{i,p})])$, which by Proposition 2.5.3 is essentially the discriminant, D_i, of the associated order Λ_i.

Finally, we write $(PB^\times \times PB^\times)(\mathbb{A}_F)^+$ for the image of $\left(B^{(1)} \times B^{(1)}\right)(\mathbb{A}_F)$ under the natural projection map (see Definition 3.3.7).

Theorem 2.6.5. *Let μ_i be a sequence of probability measures on*

$$(PB^\times(F) \setminus PB^\times(\mathbb{A}_F))^2$$

associated to the homogeneous joint toral sets $[\mathcal{T}_i(g_i, s_i g_i)]$. Assume the following

1. *The discriminants $|D_i| \to \infty$, and the tori $\mathbb{T}_i$ split at all places in S.*

2. *The local discriminants at S are bounded, i.e.*

$$\forall p \in S, \mathrm{disc}_p([\mathcal{T}_i l_i]) \ll 1$$

3. *The conductors are bounded*

$$\mathfrak{f}_i = \mathfrak{f}(\Lambda_i) \ll 1$$

4. *There are no Siegel zeroes. More specifically, there exists $C > 0$ satisfying*

$$\frac{L'(1, \chi_{E_i})}{L(1, \chi_{E_i})} \leq C \log |D_{E_i}|,$$

5. *For $\chi_j^{(i)}$ running over the characters of $\mathbb{A}_{E_i}^{\times}$ that are trivial on $\mathcal{T}_i$ (all of which are quadratic),*

$$\sum_{j:\chi_j^{(i)} \neq 1} L(\chi_j^{(i)}, 1) \ll \log |D_{E_i}|, \text{ as } i \to \infty.$$

6. *There exists $\eta > 0$ such that*

$$\min_{\substack{\mathfrak{a} \subset \Lambda_i \\ [\mathfrak{a}] = [\mathfrak{s}_i]}} N\mathfrak{a} \geq |D_{\Lambda_i}|^{\frac{1}{2(2d+1)} + \eta},$$

and a non-trivial bound on exponential sums on $(\mathcal{O}_F/\mathcal{I})^2$ (as in Assumption 7.1.1 of Section 7.1 with the sums defined in equation (7.3) of that Section), i.e. there exists $\theta > 0$ such that

$$|G_{w,a,\mathcal{I},\mathfrak{r}}| \leq (N\mathcal{I}^2)^{1-\theta}.$$

Then any weak- limit of the sequence $\{\mu_i\}$ is $(PB^{\times} \times PB^{\times})(\mathbb{A}_F)^+$-invariant.*

The condition (5) is a rather strict condition on the subset $\mathcal{T}_i < \mathbb{T}_i(\mathbb{A}_F)$, and one that we hope to remove in future, however for now we simply note that it is trivially satisfied when we take full torus orbits.

Chapter 3

Ergodic Theory

In this section, we review ergodic theory and its applications to the study of locally homogeneous spaces and equidistribution. In particular, we provide the general recipe for reducing arithmetic questions of equidistribution of torus orbits to the analytic question of decay of certain correlations.

These correlations can then be analysed in each specific case, often using automorphic information such as Fourier coefficients or subconvexity of automorphic L-functions. This is the work of the rest of the thesis.

There are two reasons that we include a careful analysis of the ergodic theoretic background despite the fact that many of the results appear in a similar way in the literature. The first is that we intend to apply the same recipe for proving equidistribution in a variety of settings in later work, and so a general treatment here demonstrates the feasibility of that goal. Also, our intention is to allow the problem of equidistribution to be picked up by other mathematicians with more knowledge of the analytic side, without the burden of having to learn the necessary ergodic background.

The second is to understand the limitations of the procedure for analysing the action of subgroups of the class group in the arithmetic setting - it is not always the case that we would like to understand the action of an entire S-adic torus, but perhaps a suitably large subgroup of it. We hope to make it clear here that in certain cases no essential change to the method is required to achieve these results.

3.1 Measure Theory

The underlying structure for all of our results is that of a measure space.

Definition 3.1.1. *A **measure space** $(X, \mathscr{B}, \mu)$ is a set X with a σ-algebra $\mathscr{B}$ of subsets of X and a measure μ defined on $\mathscr{B}$. The measure space is called a **probability space** if $\mu(X) = 1$. We denote by $\mathcal{M}(X) = \mathcal{M}(X, \mathscr{B})$ the set of probability measures on $(X, \mathscr{B})$.*

There are many ways of producing new measures from old ones - here we will consider two in detail. One is quite general, that of conditional measures, and one is specific to the case of A-invariant measures on a space with a $\mathbb{G}(\mathbb{Q}_S)$-action, that of leafwise measures. Since leafwise measures are used to compute entropy on locally homogeneous spaces, they are very important for our situation, and they are defined via conditional measures, so we must briefly recall conditional expectation and measures here.

Proposition 3.1.2 ([EL08], §4)**.** *Let $(X, \mathscr{B}, \mu)$ be a probability space, and $\mathscr{A} \subset \mathscr{B}$ a sub-σ-algebra. Then there exists a unique continuous linear functional*

$$E_\mu(\cdot|\mathscr{A}) : L^1(X, \mathscr{B}, \mu) \to L^1(X, \mathscr{A}, \mu)$$

*called the **conditional expectation** of f given $\mathscr{A}$, satisfying*

1. *for any $f \in L^1(X, \mathscr{B}, \mu)$, the function $E_\mu(f|\mathscr{A})$ is $\mathscr{A}$-measurable; and*
2. *for all $A \in \mathscr{A}$ and $f \in L^1(X, \mathscr{B}, \mu)$,*

$$\int_A E_\mu(f|\mathscr{A}) d\mu = \int_A f d\mu.$$

We imagine that the function $E_\mu(f|\mathscr{A})$ computes the average over each atom $[x]_\mathscr{A}$ of $\mathscr{A}$ and then is the constant function on each atom with value equal to the average. (Recall that the atom $[x]_\mathscr{A}$ is defined to be the smallest element of $\mathscr{A}$ that contains x, a notion which is only well-defined for countably generated σ-algebras.) This intuition is correct for countably generated $\mathscr{A}$, but for non-countably generated $\mathscr{A}$ there is no notion of atom to speak of. However, the intuition of integrating over the atom can be made more precise by the notion of conditional measures.

Proposition 3.1.3 ([EL08], §5)**.** *Let X be a locally compact, second countable metric space, and $\mathscr{B}$ be the Borel σ-algebra on X. Let μ be a probability measure on $(X, \mathscr{B})$, and $\mathscr{A} \subset \mathscr{B}$ be a sub-σ-algebra. Then there exists a subset $X' \in \mathscr{A}$ of full measure* *and almost everywhere unique*[1] *Borel probability measures* $\mu_x^{\mathscr{A}}$ *for* $x \in X'$*, called the* ***conditional measures****, such that for every* $f \in L^1(X, \mathscr{B}, \mu)$*, we have equality*

$$E_\mu(f|\mathscr{A})(\bullet) = \int_{y \in X} f(y) d\mu_\bullet^{\mathscr{A}}(y) \in L^1(X, \mathscr{A}, \mu).$$

If $\mathscr{A}$ *is countably generated, the map* $x \mapsto \mu_x^{\mathscr{A}}$ *is constant on each atom of* $\mathscr{A}$ *contained in* X'*, and* $\mu_x^{\mathscr{A}}$ *is supported on* $[x]_{\mathscr{A}}$*.*

3.1.1 Invariant Measures and Ergodicity

One of the major ideas in the field, essentially due to Linnik, is that in many arithmetic scenarios, particularly when group actions are involved, measure theory can often interact nicely with the algebraic action, resulting in non-trivial analytic information. Recall (e.g. from [EW11]) the basic set-up of measure preserving transformations.

Definition 3.1.4. *Let* $(X, \mathscr{B}, \mu)$ *and* $(Y, \mathscr{C}, \nu)$ *be probability spaces. A map* $T : X \to Y$ *is* ***measurable*** *if* $T^{-1}(A) \in \mathscr{B}$ *for all* $A \in \mathscr{C}$*. It is called* ***measure preserving*** *if it is measurable and* $\mu(T^{-1}A) = \nu(A)$ *for all* $A \in \mathscr{C}$*. If* $(Y, \mathscr{C}, \nu) = (X, \mathscr{B}, \mu)$ *then we often say that* μ *is a* ***T-invariant measure****. For* $T : X \to X$ *measurable, we denote by* $\mathcal{M}(X)^T$ *the set of* T*-invariant probability measures.*

Suppose that G is a (σ-locally compact[2]) metric group acting on a compact metric space, X, which we endow with the Borel σ-algebra, such that the action

$$\rho : G \times X \to X$$

is continuous, then for a measure $\mu \in \mathcal{M}(X)$ we say the action of G on X is **measure-preserving** if $\rho(g, -) : X \to X$ is measure-preserving for all $g \in G$. We also say μ is a **G-invariant measure**. The set of G-invariant measures is denoted $\mathcal{M}(X)^G$. We say that an orbit of G is **periodic** if it supports a finite G-invariant measure.

If the group G has algebraic structure, and X is related to this in some precise way (see below), then there is a particular class of measures which are distinguished by this algebraic structure. Let $\mathbb{G}$ be a linear algebraic group defined over $\mathbb{Q}$, S a finite set of places of $\mathbb{Q}$, and $\Gamma < \mathbb{G}(\mathbb{Q}_S)$ a lattice[3]. The definition we use is from [Kha19a, §4]

[1] Meaning two different choices $\{\mu_x^{\mathscr{A}}\}_{x \in X'}, \{\widetilde{\mu}_x^{\mathscr{A}}\}_{x \in X'}$, then the set $\{x \in X' : \mu_x^{\mathscr{A}} \neq \widetilde{\mu}_x^{\mathscr{A}}\}$ is a null set.

[2] See [EW11] - this is short hand for locally compact and a union of countably many compact subspaces.

[3] A lattice in a locally compact group G is a discrete subgroup, Γ, such that the quotient space, $\Gamma \setminus G$ has a finite G-invariant measure.

Definition 3.1.5. *1. A probability measure ν on $X = \Gamma \setminus \mathbb{G}(\mathbb{Q}_S)$ is* ***algebraic*** *if there is a closed unimodular algebraic subgroup $\mathbb{L} < \mathbb{G}$ defined and anisotropic over $\mathbb{Q}$, a finite index subgroup $L < \mathbb{L}(\mathbb{Q}_S)$ and some $g_S \in \mathbb{G}(\mathbb{Q}_S)$ such that ν is the unique L-Haar probability measure*[4] *on $[Lg_S] \subset \Gamma \setminus \mathbb{G}(\mathbb{Q}_S)$.*

2. A probability measure ν on $\mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A})$ is ***algebraic*** *if there is a closed unimodular subgroup $\mathbb{L} < \mathbb{G}$ defined and anisotropic over $\mathbb{Q}$, an isogeny $\mathbb{L}' \to \mathbb{L}$ over $\mathbb{Q}$ and a closed subgroup of finite index $L < \mathrm{Im}(\mathbb{L}'(\mathbb{A}) \to \mathbb{L}(\mathbb{A}))$ and some $g \in \mathbb{G}(\mathbb{A})$ such that ν is the L-Haar probability measure on $[Lg] \subset \mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A})$.*

In both cases, the orbit $[Lg]$ is a periodic $g^{-1}Lg$-orbit, which we call an ***$\mathbb{L}$-homogeneous set****. If $\mathbb{L}$ is a maximal torus, we call this a* ***homogeneous toral set****.*

It's clear that if we have sets $S' \subset S$ such that $S \setminus S'$ consists of finite places, and $K \subset \mathbb{G}(\mathbb{Q}_{S \setminus S'})$ an open compact subgroup, then an algebraic measure on $\Gamma \setminus \mathbb{G}(\mathbb{Q}_S)$ projects, under the quotient by K, to a sum of algebraic measures on $\Gamma \cap K \setminus \mathbb{G}(\mathbb{Q}_{S'})$. The same is true for projecting from the adelic version of an algebraic measure to the S-adic version. As we will see later, it is possible under certain conditions to go the other way.

We now move onto a crucial property of measures, ergodicity, which characterises the building blocks of invariant measures. We will make heavy use of hard classification theorems of ergodic measures. The notation $A \triangle B := (A \setminus B) \cup (B \setminus A)$ refers to the symmetric difference.

Definition 3.1.6. *A measure preserving group action of G on X is* ***ergodic*** *if any set $A \in \mathscr{B}$ such that $\mu(g^{-1}A \triangle A) = 0$ for all $g \in G$ has $\mu(A) \in \{0, 1\}$. We say the measure μ is* ***G-invariant and ergodic****.*[5] *We denote the set of such measures by $\mathcal{E}(X)^G$.*

Since we have assumed that the space X is compact, it is possible to put a metric on the set $\mathcal{M}(X)$ such that the convergence of measures in this metric coincides with the notion of weak* convergence, that is $\mu_i \to \mu$ if and only if for every $f \in \mathcal{C}(X, \mathbb{R})$,

$$\lim_{i \to \infty} \int_X f d\mu_i \to \int_X f d\mu.$$

[4] To explain precisely what we mean by this, notice that $[Lg_S]$ has a natural bijection with $\Gamma \cap L \setminus L$. Both L and $\Gamma \cap L$ are unimodular, and so there is an L-invariant measure on the quotient. Since $\mathbb{L}$ is anisotropic, this measure is finite, so can be normalised to be a probability measure, and we are stating that ν is this measure pulled back along the above bijection.

[5] Often the expression "μ is G-invariant and ergodic" is used, and it should be kept in mind that both invariance and ergodicity depend on G, so this is more accurately described as "G-invariant and G-ergodic".

With this topology, $\mathcal{M}(X)$ is a compact and convex metric subspace of a real vector space (since we can add and scale measures). For any measurable transformation $T : X \to X$, the subspace $\mathcal{M}(X)^T$ is closed and also convex. Therefore we see that

$$\mathcal{M}(X)^G := \bigcap_{g \in G} \mathcal{M}(X)^{\rho(g,-)}$$

is a closed (therefore compact), convex subspace of $\mathcal{M}(X)$. By applying Choquet's Theorem, we would expect that an arbitrary G-invariant measure to be uniquely decomposable into the extremal points of $\mathcal{M}(X)^G$, and in fact these extremal points are precisely the ergodic measures. That is, we have the following ergodic decomposition:

Proposition 3.1.7. *For $\mu \in \mathcal{M}(X)^G$, there exists a measure ν on $\mathcal{M}(X)^G$ such that*

1. *ν is supported on the ergodic measures, that is $\nu\left(\mathcal{E}(X)^G\right) = 1$;*
2. *μ is the ν-weighted average of G-invariant and ergodic measures, so*

$$\mu = \int_{\mathcal{M}(X)^G} \xi d\nu(\xi).$$

Sketch Proof. Let $\mathscr{E}$ be the σ-algebra of G-invariant sets. Then for any $x \in X$, the conditional measure $\mu_x^{\mathscr{E}}$ is G-invariant and ergodic, and

$$\mu = \int \mu_x^{\mathscr{E}} d\mu.$$

Thus ν is the push-forward of μ along the measurable map $x \mapsto \mu_x^{\mathscr{E}} \in \mathcal{M}(X)^G$. □

We will also consider the problem of joint equidistribution, which in the translated world of measure theory corresponds to problems about joinings of measures.

Definition 3.1.8. *Let $(X_i, \mathscr{B}_i, \mu_i)_{i=1,2}$ be a pair of probability spaces with a measure preserving action of a group G. A **joining** of these two G-invariant measures is a G-invariant probability measure ρ on $(X_1 \times X_2, \mathscr{B}_1 \otimes \mathscr{B}_2)$ such that $\pi_{i,*}\rho = \mu_i$ for $i = 1, 2$.*

3.2 Entropy

Entropy is roughly an indication of how complicated a system is - higher entropy should correspond to more complicated systems. There are three definitions of entropy that we consider in different situations, each built on the previous one:

1. The static entropy of a partition on a probability space indicates how much information one gains about a point from learning which element of the partition that point is in.
2. The dynamic entropy of a partition on a probability space with a measurable transformation indicates how quickly one gains information about a point from learning how its orbit under the transformation moves through the partition.
3. The ergodic theoretic entropy of a probability space with a measurable transformation indicates how irregular the orbits of that transformation are.

Here are the formal definitions (see §3 of [EL08]):

Definition 3.2.1. *Let (X,μ) be a probability space. The* ***static entropy*** *$H_\mu(\mathcal{P})$ of a finite or countable partition $\mathcal{P}$ of X is defined to be*

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

Definition 3.2.2. *Let (X,μ) be a probability space and $T : X \to X$ a μ-preserving map. Let $\mathcal{P}$ be a finite or countable partition of X with $H_\mu(\mathcal{P}) < \infty$. The* ***dynamic entropy*** *of $(X,\mu,T,\mathcal{P})$ is*

$$h_\mu(T,\mathcal{P}) = \lim_{N\to\infty} \frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right).$$

Here $T^{-n}\mathcal{P}$ is the partition $\{T^{-n}P : P \in \mathcal{P}\}$, and $\bigvee_i \mathcal{Q}_i$ denotes the smallest partition such that each element $Q \in \mathcal{Q}_i$ is a disjoint union of elements of $\bigvee_i \mathcal{Q}$.

Definition 3.2.3. *Let (X,μ) be a probability space and $T : X \to X$ a μ-preserving map. The* ***ergodic theoretic entropy*** *of (X,μ,T) is defined to be*

$$h_\mu(T) = \sup_{\mathcal{P}:H_\mu(\mathcal{P})<\infty} h_\mu(T,\mathcal{P}).$$

To give two simple examples, it is easy to see that if we take the Borel probability space $(\mathbb{S}^1,\mu)$ with any rotation T by a rational multiple of 2π, then the ergodic theoretic entropy is zero. The orbits are very regular and the system is not at all complicated.

On the other hand, the measure preserving map $\mathbb{S}^1 \to \mathbb{S}^1$ given by squaring has entropy $\log(2)$. It has complicated orbits, some finite and some equidistributed.

3.2.1 Entropy on Locally Homogeneous Spaces

It will be important to understand the entropy of particular measures on the locally homogeneous spaces associated with semi-simple groups. Recall the scenario in which we are working, as laid out in Section 2. One of the crucial advances in the understanding of measures on the space $X = \Gamma \setminus \mathbb{G}(\mathbb{Q}_S)$ comes from the realisation of Einsiedler, Katok and Lindenstrauss that the entropy of measures on X can be split into contributions coming from the roots of $\mathbb{G}(\mathbb{Q}_S)$.

Let Φ denote the set of (non-zero) roots of G^{spl}, which is the split $\mathbb{Q}$-algebraic group such that G^{spl}_F is a form of G. Recall that

$$\mathbb{G}(\mathbb{Q}_S) = \prod_{p\in S} \left(G^{spl}(\mathbb{Q}_p)\right)^d, \tag{3.1}$$

where $d = [F : \mathbb{Q}]$, by our assumption that F splits completely over p, and G splits at all places over p in S, and the bijection fixed in point 6 of Section 1.3. The group $A \leq \mathbb{G}(\mathbb{Q}_S)$ is a product of (finite index subgroups of) maximal split tori in each of the factors.

Definition 3.2.4. *A prime $\nu|p \in S$ of F along with a root $\alpha \in \Phi$ defines a group homomorphism, α_ν, of A to $\mathbb{R}_{>0}$ via*

$$\alpha_\nu((a_{\nu'})_{\nu'|S}) = \log|\alpha(a_\nu)|_\nu,$$

*called the **Lyapunov root** associated to α and ν. In other words, the bijection of (3.1) associates to each prime ν dividing a prime $p \in S$ precisely one of the d factors in the p-summand, and then the Lyapunov root is the projection to this factor followed by the usual (additive) character associated to $\alpha \in \Phi$.*

Thus the collection of all Lyapunov roots for $\mathbb{G}(\mathbb{Q}_S)$ is equal to $\Phi_{\text{Lpv}} := \Sigma_{F,S} \times \Phi$. Given $\alpha_\nu \in \Phi_{\text{Lpv}}$, we consider the root α of $G^{spl}(\mathbb{Q}_p)$, which gives rise to a nilpotent Lie subalgebra $\mathfrak{u}^{\alpha_\nu} \subset \mathfrak{g}_\nu$ and an abelian one dimensional unipotent subgroup $\exp(\mathfrak{u}^{\alpha_\nu}) = U^{\alpha_\nu} \leq G(F_\nu) = G^{spl}(\mathbb{Q}_p)$ which is normalised by A.[6]

Definition 3.2.5. *For an element $a \in A$, define the **contracted roots** of a to be*

$$\Phi_a^- := \{\alpha_\nu \in \Phi_{\text{Lpv}} : \alpha_\nu(a) < 0\}.$$

[6]Notice that in our case, as opposed to [Ein+15], the S-adic group $\mathbb{G}(\mathbb{Q}_S)$ is split and so we don't need to worry about the possibility of non-irreducible roots, and everything works as though over an algebraically closed field. In particular, the unipotent subgroups are one-dimensional.

For any closed subset[7] $\Psi \subset \Phi_a^-$ *of contracted roots, define*

$$\mathfrak{u}^{\Psi} = \sum_{\alpha^\nu \in \Psi} \mathfrak{u}^{\alpha_\nu}.$$

We define the ***stable horospherical subgroup***

$$U_a := \langle U^{\alpha_\nu} : \alpha_\nu \in \Phi_a^- \rangle = \exp \mathfrak{u}^{\Phi_a^-},$$

and the groups $U^{\Psi} = \exp \mathfrak{u}^{\Psi}$ *for* $\Psi \subset \Phi_a^-$ *closed are called the* ***connected*** a***-stable unipotent subgroups*** *of* $\mathbb{G}(\mathbb{Q}_S)$.

Example 3.2.6. *We give an example of the above facts, just to clarify the ideas. Suppose that* $S = \{p, q\}$ *for* $p \neq q$*, and* $G = \mathrm{PGL}_{3/\mathbb{Q}}$. *Let* A *be the diagonal matrices, and* $a = \left(\begin{pmatrix} 1 & & \\ & 3 & \\ & & 2 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \right) \in \mathrm{PGL}_3(\mathbb{Q}_{pq})$. *Label the non-zero roots of* PGL_3 *by* $\alpha^{(ij)}$ *for* $1 \leq i \neq j \leq 3$ *in the usual way. The set of contracted roots is*

$$\Phi_a^- = \left\{ \alpha_p^{(12)}, \alpha_p^{(13)}, \alpha_p^{(32)}, \alpha_q^{(12)}, \alpha_q^{(13)} \right\}.$$

Any subset of these forms a closed subset, and the stable horospherical subgroup is the subgroup

$$U_a = \left\{ \left(\begin{pmatrix} 1 & x & y \\ & 1 & \\ & z & 1 \end{pmatrix}, \begin{pmatrix} 1 & s & t \\ & 1 & \\ & & 1 \end{pmatrix} \right) : x, y, z \in \mathbb{Q}_p, s, t \in \mathbb{Q}_q \right\}.$$

To define the contribution to entropy that comes from a Lyapunov root, we need to define leafwise measures. These are similar to conditional measures, but split an A-invariant measure up over the orbits of a unipotent group U normalised by A.

Definition 3.2.7. *Let* μ *be an* A*-invariant probability measure on* X. *For any unipotent subgroup* $U \subset U_a$ *which is normalised by* A*, there is a system* $\left\{ \mu_x^U \right\}_{x \in X}$ *of Radon measures*[8] *on* U*, called the* ***leafwise measures*** *for* U*, and a* μ*-co-null set* $X' \subset X$ *with the following properties:*

1. *The map* $x \mapsto \mu_x^U$ *is measurable.*

2. *For every* $\epsilon > 0$ *and* $x \in X'$*, we have* $\mu_x^U(B_\epsilon^U) > 0$.

[7]Meaning $\Psi + \Psi \cap \Phi_a^- \subset \Psi$.

[8]Radon measures are measures on the σ-algebra of Borel sets such that for every open set U, $m(U) = \sup_{\text{cpt } K \subset U} m(K)$ and every point has a neighbourhood with finite measure.

3. *For every* $x \in X'$ *and* $u \in U$ *with* $u \cdot x \in X'$, *we have that* $\mu_x^U \propto u_* \left(\mu_{u \cdot x}^U\right)$.

4. *For every* $a \in A$, *and* $x, a \cdot x \in X'$, *we have that* $\mu_{a \cdot x}^U \propto (\mathrm{Ad}_a)_* \left(\mu_x^U\right)$.

5. *Suppose* $Z \subset X$ *and that there exists a countably generated* σ*-algebra* $\mathcal{A}$ *of subsets of* Z *such that for any* $x \in Z$,

$$U_{x,\mathcal{A}} := \{t : t \cdot x \in [x]_{\mathcal{A}}\}$$

is open and bounded in U, *and* $[x]_{\mathcal{A}} = U_{x,\mathcal{A}} \cdot x$, *i.e. atoms of* $\mathcal{A}$ *are orbits of open pre-compact subsets of* U. *Then the conditional measures of* $\mu|_Z$ *with respect to* $\mathcal{A}$ *are given by pushforwards of the measures* μ_x^U, *i.e. for* μ*-almost every* $x \in Z$,

$$(\mu_Z)_x^{\mathcal{A}} \propto x_* \left(\mu_x^U|_{U_{x,\mathcal{A}}}\right).$$

We should imagine that given a point $x \in X$, the restriction of μ to the orbit of x is equal to the pushforward of the leafwise measure at x under the map $U \to X, u \mapsto u \cdot x$. This is not true exactly, but it is the right intuition, as described by point (5) above.

The entropy of the action of a with respect to the measure μ can be calculated using the leafwise measures for the stable horospherical subgroup U_a, which can then be decomposed into entropy contributions coming from each of the contracted roots of a. This allows us to compute precisely the entropy of the Haar measure, as well as the entropy of other algebraic measures on X, and giving a relationship between the entropy of the limit of torus orbits and the decay of volume of Bowen balls, as will be explained in Section 3.5.

Definition 3.2.8. *Let* U *be a connected* a*-stable unipotent subgroup of* U_a, *we define the* U***-density*** *at* x *as*

$$D_\mu(a, U) := -\lim_{n \to \infty} \frac{\log \mu_x^U \left(\mathrm{Ad}_a^n B_1^U\right)}{n}.$$

This measures how concentrated the leafwise measure for U *is at* x. *The* ***entropy contribution*** *of* U *is*

$$h_\mu(a, U) := \int_X D_\mu(a, U) d\mu.$$

The main utility of these entropy contributions is the fact that the entropy of a is precisely the entropy contribution of its stable horospherical subgroup, and that this breaks up into a sum of the entropy contributions of each of the contracted roots.

Theorem 3.2.9 ([EK05], Prop 9.4, [EL08], Cor 9.10)**.**

$$h_\mu(a) = h_\mu(a, U_a) = \sum_{\alpha_\nu \in \Phi_a^-} h_\mu(a, U^{\alpha_\nu}).$$[9]

We now use this to prove a criterion for a lower bound on the entropy based on asymptotic volumes of specific subsets called Bowen balls, which are closely related to the sets $\mathrm{Ad}_a^n B_1^U$ arising in the definition of the U-density in Definition 3.2.8.

Definition 3.2.10. *Let $n \geq 0$, and $B \subset \mathbb{G}(\mathbb{Q}_S)$ (resp. $\mathbb{G}(\mathbb{A})$) be an open neighbourhood of the identity. Then we define the n-**Bowen ball** to be*

$$B^{(n)} := B_a^{(n)} = \bigcap_{r=-n}^{n} \mathrm{Ad}_a^r B,$$

*and the n-**Bowen ball at** $x \in X$ (resp. $x \in [\mathbb{G}(\mathbb{A})]$) to be $xB^{(n)}$. Note that $a \in \mathbb{G}(\mathbb{Q}_S)$ and so the conjugation action of a is trivial on places of $\mathbb{G}(\mathbb{A})$ outside S.*

A crucial quantity that will recur throughout this thesis is that of correlation between measures, previously studied in [Kha17; Kha19a; Kha19b].

Definition 3.2.11. *For two probability measures μ, ν and a compactly supported bounded measurable function f on $[\mathbb{G}(\mathbb{A})]$, define the **kernel function** by*

$$K_f(x, y) = \sum_{\gamma \in \mathbb{G}(\mathbb{Q})} f(x^{-1}\gamma y),$$

*which is a bounded function on $[\mathbb{G}(\mathbb{A})]^2$. From this, we define the **correlation between** μ **and** ν by*

$$\mathrm{Corr}(\mu, \nu)[f] = \int \int K_f(x, y) d\mu(x) d\nu(y).$$

We write $\mathrm{Corr}(\mu, \nu)[B]$ when $f = 1_B$ is the indicator function of a compact subset $B \subset [\mathbb{G}(\mathbb{A})]$.

The next proposition is Proposition 3.2 of [Ein+06], and we refer to that paper for the complete proof. We give a sketch proof here via the characterisation of entropy with leafwise measures. The idea of the proof is simply that the volume of the Bowen ball shrinks in the direction of the stable horospherical subgroup and its opposite unipotent subgroup at a rate proportional to the U-density. Thus if the volume of Bowen balls shrink rapidly, we expect the entropy to be large.

[9]Note that this theorem is slightly simpler than the general case since we have no reducible Lyapunov roots, and so each of the connected a-stable unipotent subgroups U^{α_ν} is one-dimensional.

Proposition 3.2.12. *Let $a \in A$, and μ be an a-invariant probability measure on X. Suppose that there exists $B \subset \mathbb{G}(\mathbb{Q}_S)$ an open neighbourhood of the identity such that*

$$\mathrm{Corr}(\mu,\mu)[B^{(n)}] \ll e^{-2\eta n} \text{ as } n \to \infty,$$

then $h_\mu(a) \geq \eta$.

Sketch. Suppose we normalise the leafwise measures to satisfy $\mu_x^{U_{\alpha\nu}}(B_1^{U_{\alpha\nu}}) = 1$. The definition of $\mathrm{Corr}(\mu,\mu)\left[B^{(n)}\right]$ shows that, up to a multiplicative constant bounded in terms of n,

$$\mathrm{Corr}(\mu,\mu)\left[B^{(n)}\right] = \int_X \mu\left(xB^{(n)}\right) d\mu(x).$$

Combining Theorem 3.2.9 and Definition 3.2.8, we see that

$$h_\mu(a) = -\frac{1}{2}\int_X \lim_{n\to\infty}\frac{1}{n}\log\left(\prod_{\Phi_a^-}\mu_x^{U_{\alpha\nu}}\left(\mathrm{Ad}_a^n B_1^{U_{\alpha\nu}}\right)\prod_{\Phi_a^+}\mu_x^{U_{\alpha\nu}}\left(\mathrm{Ad}_a^{-n} B_1^{U_{\alpha\nu}}\right)\right) d\mu(x)$$

Then, for a fixed point $x \in X$, up to a bounded multiplicative constant, the volume of $xB^{(n)}$ is equal to

$$\prod_{\Phi_a^-}\mu_x^{U_{\alpha\nu}}\left(\mathrm{Ad}_a^n B_1^{U_{\alpha\nu}}\right)\prod_{\Phi_a^+}\mu_x^{U_{\alpha\nu}}\left(\mathrm{Ad}_a^{-n} B_1^{U_{\alpha\nu}}\right).$$

This is because $B^{(n)}$ is a rectangle which is contracted according to Ad_a in the stable directions for a, and is contracted according to Ad_a^{-1} in the unstable directions for a. Therefore,

$$\begin{aligned} h_\mu(a) &= -\frac{1}{2}\int_X \lim_{n\to\infty}\frac{1}{n}\log\left(\mu\left(xB^{(n)}\right)\right) d\mu(x) \\ &= -\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\int_X \log\left(\mu\left(xB^{(n)}\right)\right) d\mu(x) \\ &\geq -\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\log \mathrm{Corr}(\mu,\mu)\left[B^{(n)}\right] \\ &\geq \eta \end{aligned}$$

where the penultimate inequality is Jensen's inequality, and we can switch the integral and the limit by dominated convergence since we are assuming that X is a compact space. □

We see that upper bounds on the self-correlation leads to lower bounds on the entropy. Giving lower bounds on the entropy is crucial to applying the measure theoretic results of the next section. In particular, we should imagine that the higher the entropy, the more well-behaved a measure becomes, until the point where high enough entropy implies the measure must essentially be the Haar measure. The strategy of proof for the joint CM case is to show that the entropy is larger than that of any measure constrained to an intermediate subvariety.

3.3 Measure Theoretic Classifications

Here we recall the work of Einsiedler-Lindenstrauss in [EL15].

Theorem 3.3.1. *Let $\mathbb{G}$ be a semisimple linear algebraic group over $\mathbb{Q}$, $X = \mathbb{G}(\mathbb{Q}) \cap \mathcal{K}^S \setminus \mathbb{G}(\mathbb{Q}_S)$, and A a subgroup of finite index inside the group of $\mathbb{Q}_S$-points of an S-split maximal torus defined over $\mathbb{Q}_S$. Let μ be an A-invariant and ergodic probability measure on X, and let $p \in \mathbb{G}(\mathbb{Q}_S)$ be such that $\Gamma p \in \operatorname{supp}\mu$ (where $\Gamma = \mathbb{G}(\mathbb{Q}) \cap \mathcal{K}^S$). Then there exists a reductive linear algebraic subgroup $\mathbb{L} \leq \mathbb{G}$ defined over $\mathbb{Q}$ with equal $\mathbb{Q}_S$-rank to $\mathbb{G}$ (i.e. $\mathbb{L}$ has the same geometric rank as $\mathbb{G}$ and is split over $\mathbb{Q}_S$) such that:*

(S) The measure μ is supported on the periodic orbit $\Gamma\mathbb{L}(\mathbb{Q}_S)p$, and $\mathbb{L}$ is the smallest $\mathbb{Q}$-group so that some right translate of $\Gamma\mathbb{L}(\mathbb{Q}_S)$ supports μ.

(D) $\mathbb{L}$ is the almost direct product of a $\mathbb{Q}$-anisotropic $\mathbb{Q}$-torus $\mathbb{L}_T$ and semisimple algebraic $\mathbb{Q}$-subgroups $\mathbb{L}_I, \mathbb{L}_R, \mathbb{L}_Z$. Furthermore, if we set for $t \in \{T, I, R, Z\}$ the group A_t to be $A \cap p^{-1}\mathbb{L}_t(\mathbb{Q}_S)p$, then

$$\breve{A} := A_T A_I A_R A_Z \leq A$$

has finite index.

(T) The quotient $(\mathbb{L}_T(\mathbb{Q}_S) \cap \Gamma)\backslash\mathbb{L}_T(\mathbb{Q}_S)$ is a compact abelian group, and there exists a closed subgroup $T \subset \mathbb{L}_T(\mathbb{Q}_S)$ containing pA_Tp^{-1} so that μ is $p^{-1}Tp$-invariant and $(T \cap \Gamma)/T$ is compact.

(I) There exists a finite index subgroup $L_I < p^{-1}\mathbb{L}_I(\mathbb{Q}_S)p$ that is normalized by A_I such that μ is L_I-invariant and for μ-almost every x, the orbit xL_I is periodic.

(R) The algebraic subgroup $\mathbb{L}_R$ is an almost direct product $\mathbb{L}_R = \prod_i \mathbb{L}_{R,i}$ of $\mathbb{Q}$-almost simple algebraic groups, and A_R contains the product of the subgroups $A_{R,i} = A \cap p^{-1}\mathbb{L}_{R,i}(\mathbb{Q}_S)p$ as a finite-index subgroup and $\operatorname{rank}(A_{R,i}) = 1$ for all i.

(Z) $h_\mu(a) = 0$ *for all* $a \in A_Z$*.*

This means that the possibilities for ergodic measures invariant under tori are fairly restricted by the algebraic structure of the group $\mathbb{G}$, especially if the relevant algebraic subgroups $\mathbb{L} \leq \mathbb{G}$ are simple.

While we do not directly use this Theorem in the thesis, we state it in its full form here because this is a crucial barrier to completing the cases of single equidistribution, and therefore we wish to add a short discussion here to highlight this obstruction. When proving a joint equidistribution theorem, it is certainly necessary to know single equidistribution (by projecting onto either coordinate), and in fact the present method of proving joint equidistribution requires the knowledge of single equidistribution (see the next section on the classification joinings).

In current cases where single equidistribution is known via an ergodic method, the result above is key. In rank 1 situations this theorem is extremely effective, since it implies that as soon as the entropy is positive, the measure must be algebraic (since toral measures cannot have positive entropy). More generally, the theorem simplifies under the following:

Assumption 3.3.2. *Suppose that every reductive* $\mathbb{Q}$*-anisotropic subgroup of* $\mathbb{G}$ *with equal rank to* $\mathbb{G}$ *is simple.*

When Assumption 3.3.2 holds, the decomposition in Theorem 3.3.1 simplifies particularly when $h_\mu(a) > 0$ for some $a \in A$. With this positive entropy assumption, we see that the measure μ must be essentially the pushforward of the Haar measure from $\mathbb{L}$ to $\Gamma\mathbb{L}(\mathbb{Q}_S)p$, in other words the measure μ must be algebraic. Even here however, the algebraic measures on these intermediate subgroups do have positive entropy, and are hard to rule out without very good bounds on the correlations of tori, which are currently out of reach. The possible existence of non-algebraic periodic measures in the general case is an extremely serious obstacle, which appears to be even more intractable.

Another method of trying to prove equidistribution relies on proving that $h_\mu(a)$ is very close to $h_{Haar}(a)$ and then using a measure rigidity theorem (which says that there is a unique measure with near maximal entropy, the Haar measure). It seems that this method may be the only way to approach the single equidistribution of the unitary groups.

3.3.1 Classification of Joinings

Another crucial result of Einsiedler and Lindenstrauss is on joinings of Haar measures on homogeneous spaces. Before stating the result, we require some definitions, and we add some discussion on when these definitions apply.

Definition 3.3.3. *Let X be an S-arithmetic quotient of a perfect (i.e. equal to its commutator subgroup) connected linear algebraic group $\mathbb{G}$ (which we will later take to be $\mathbb{P}$ or $\mathbb{G} \times \mathbb{G}$). We say that X is saturated by unipotents if the group generated by all unipotent elements of $\mathbb{G}(\mathbb{Q}_S)$ acts ergodically on X.*

Definition 3.3.4. *A subgroup $A < \mathbb{G}(\mathbb{Q}_S)$ is said to be of class-$\mathcal{A}'$ if it can be simultaneously diagonalised (i.e. it is semi-simple and abelian) so that for every $a \in A$ and each $p \in S$, the projection of a to $\mathbb{G}(\mathbb{Q}_p)$ has eigenvalues which are all powers of a fixed $\lambda_p \in \mathbb{Q}_p^*$ with $|\lambda_p|_p \neq 1$.*

A homomorphism $a : \mathbb{Z}^d \to \mathbb{G}(\mathbb{Q}_S)$ is said to be of class-$\mathcal{A}'$ if it is a proper map and its image $a(\mathbb{Z}^d)$ is of class-$\mathcal{A}'$.

The following is the main result of [EL17] (Theorem 1.4 in *loc. cit.*).

Theorem 3.3.5. *Let $r, d \geq 2$, $\mathbb{G}_1, ..., \mathbb{G}_r$ be semisimple algebraic groups defined over $\mathbb{Q}$ that are $\mathbb{Q}$-almost simple, and $\mathbb{G} = \prod_{i=1}^r \mathbb{G}_i$. Let $X_i = \Gamma_i \setminus \mathbb{G}_i(\mathbb{Q}_S)$ be S-arithmetic quotients saturated by unipotents, and let $X = \prod_i X_i$. Let $a_i : \mathbb{Z}^d \to \mathbb{G}_i(\mathbb{Q}_S)$ be proper homomorphisms so that $a = (a_1, ..., a_r) : \mathbb{Z}^d \to \mathbb{G}(\mathbb{Q}_S)$ is of class-$\mathcal{A}'$, and set $\mathscr{A} = a(\mathbb{Z}^d)$. Suppose that μ is an $\mathscr{A}$-invariant and ergodic joining of the actions of $\mathscr{A}_i = a_i(\mathbb{Z}^d)$ on (X_i, m_{X_i}). Then μ is an algebraic measure defined over $\mathbb{Q}$.*

There is another important theorem that we need (Theorem 1.6 in *loc. cit.*).

Theorem 3.3.6. *Let $r \geq 1$, $d \geq 2$, and $\mathbb{G}_1, ..., \mathbb{G}_r$ be semisimple algebraic groups defined over $\mathbb{Q}$ that are $\mathbb{Q}$-almost simple, $\mathbb{G} = \prod_{i=1}^r \mathbb{G}_i$. Let $\mathbb{V}$ be a linear $\mathbb{Q}$-representation of $\mathbb{G}$ that doesn't contain the trivial representation, and set $\mathbb{P} = \mathbb{G} \ltimes \mathbb{V}$. Let $X_i = \Gamma_i \setminus \mathbb{G}_i(\mathbb{Q}_S), W = \Lambda \setminus \mathbb{V}(\mathbb{Q}_S)$ be S-arithmetic quotients which are saturated by unipotents. Let $a : \mathbb{Z}^d \to \mathbb{P}(\mathbb{Q}_S)$ be a class-$\mathcal{A}'$ map, and set $A = a(\mathbb{Z}^d)$. Then any A-invariant and ergodic measure μ on $\prod_{i=1}^r X_i \times W$ which projects to the Haar measure on $\prod_{i=1}^r X_i$ is an algebraic measure defined over $\mathbb{Q}$.*

We now give a discussion on the situations in which the conditions of properness, saturation by unipotents, and class-$\mathcal{A}'$ hold, as well as a standard gluing procedure to deduce an adelic statement. We also consider certain reductions that can be performed, e.g. to assume that $\mathbb{V}$ is irreducible (or zero). One obstacle to applying these theorems directly to our situation is that the S-arithmetic quotients we are considering may not be saturated by unipotents, since the group generated by all unipotents may be a finite index subgroup. However, we can apply the theorem to each component of X which looks like an S-arithmetic quotient of the group $\mathbb{G}^{sc}$, as discussed below.

Class-$\mathcal{A}'$ subgroups of tori

Suppose that A is a subgroup of finite index inside the group of $\mathbb{Q}_S$-points of an S-split maximal torus in $\mathbb{G}(\mathbb{Q}_S)$. How does invariance under A relate to the class-$\mathcal{A}'$ subgroups that are required in the joinings classifications?

By construction, A is finite index subgroup inside a split maximal torus of $\mathbb{G}(\mathbb{Q}_S)$, and therefore taking eigenvalues identifies it with a finite index subgroup of $\left(\mathbb{Q}_S^\times\right)^{\text{rank}(\mathbb{G})}$ (there may be more eigenvalues, but they will be monomial functions of this set of 'free' eigenvalues, for example in SL_2 the diagonal torus has two eigenvalues, but really we should think of it as having 1 after we've made a choice). Pick an element $\lambda_\nu \in \mathbb{Q}_\nu^\times$ for each place $\nu \in S$ such that $|\lambda_\nu|_\nu \neq 1$, and $\lambda_\infty > 0$ if $\infty \in S$. For each ν, we have a map

$$a_\nu : \mathbb{Z}^{\text{rank}(\mathbb{G})} \to \left(\mathbb{Q}_\nu^\times\right)^{\text{rank}(\mathbb{G})}$$

sending $(n_1, ..., n_{\text{rank}(\mathbb{G})}) \mapsto \left(\lambda_\nu^{n_1}, ..., \lambda_\nu^{n_{\text{rank}(\mathbb{G})}}\right)$. These combine to give a homomorphism $a : \mathbb{Z}^{|S|\text{rank}(\mathbb{G})} \to \left(\mathbb{Q}_S^\times\right)^{\text{rank}(\mathbb{G})}$. Some finite index subgroup of $\mathbb{Z}^{|S|\text{rank}(\mathbb{G})}$ will have image inside A, and we can relabel $a : \mathbb{Z}^{|S|\text{rank}(\mathbb{G})} \to A$ from this subgroup. Clearly, this is a class-$\mathcal{A}'$ subgroup of A such that any $\mathbb{Q}_S$-algebraic subgroup of $\mathbb{G}(\mathbb{Q}_S)$ which contains $a(\mathbb{Z}^{|S|\text{rank}(\mathbb{G})})$ must contain A.

This construction gives us the proper class-$\mathcal{A}'$ homomorphisms that we use in both the joint case and the Kuga-Sato case.

Saturation by Unipotents

Saturation by unipotents relates to the subgroup generated by all unipotents. Since saturation by unipotents is obviously guaranteed for S-adic quotients of $\mathbb{V}$, we will restrict ourselves to the semisimple group $\mathbb{G}$.

Definition 3.3.7. *Let $\mathbb{G}(\mathbb{A})^+$ be the subgroup generated by all unipotents of $\mathbb{G}(\mathbb{A})$. For any subgroup $H < \mathbb{G}(\mathbb{A})$, we define $H^+ := H \cap \mathbb{G}(\mathbb{A})^+$. Also, let $\mathbb{G}^{sc}$ denote the simply connected cover of $\mathbb{G}$, which admits a central isogeny to $\mathbb{G}$.*

See [Mar91, §1.5.2] for a more general definition. We summarise some relevant facts about this construction.

Proposition 3.3.8. *1. $\mathbb{G}(\mathbb{Q}_S)^+$ is equal to the commutator subgroup of $\mathbb{G}(\mathbb{Q}_S)$.*

2. For any $f : \mathbb{G} \to \mathbb{G}'$ a central $\mathbb{Q}$-isogeny, $f(\mathbb{G}(\mathbb{Q}_S)^+) = \mathbb{G}'(\mathbb{Q}_S)^+$.

3. $\mathbb{G}^{sc}(\mathbb{Q}_S)^+ = \mathbb{G}^{sc}(\mathbb{Q}_S)$.

4. $\mathbb{G}(\mathbb{Q}_S)^+$ is an open normal subgroup of finite index in $\mathbb{G}(\mathbb{Q}_S)$. $\mathbb{G}(\mathbb{A})^+$ is a normal subgroup of $\mathbb{G}(\mathbb{A})$, which may not be of finite index.

5. $\mathbb{G}(\mathbb{Q}_S)^+ = \mathrm{im}\,(\mathbb{G}^{sc}(\mathbb{Q}_S))$ under any central isogeny $\mathbb{G}^{sc} \to \mathbb{G}$.

Proof. The first four are contained in [Mar91]. See Theorems 1.5.6, 1.5.5 and 2.3.1. The final part follows immediately from 2 and 3. □

We see that $\Gamma \setminus \mathbb{G}(\mathbb{Q}_S)$ can be written as a disjoint union of finitely many locally homogeneous spaces for $\mathbb{G}^{sc}$, all of which are in bijection. That is, if we choose representatives ω_i for the quotient

$$\Gamma \setminus \mathbb{G}(\mathbb{Q}_S)/\mathbb{G}(\mathbb{Q}_S)^+ = \bigsqcup_j \Gamma\omega_j\mathbb{G}(\mathbb{Q}_S)^+,$$

then we can write

$$\Gamma \setminus \mathbb{G}(\mathbb{Q}_S) = \bigsqcup_j \Gamma\omega_j \cdot \mathbb{G}(\mathbb{Q}_S)^+.$$

Since $\mathbb{G}(\mathbb{Q}_S)^+ \subset \mathbb{G}(\mathbb{Q}_S)$ is normal, we get

$$\Gamma\omega_j\mathbb{G}(\mathbb{Q}_S)^+ = \Gamma \setminus \Gamma\mathbb{G}(\mathbb{Q}_S)^+\omega_j \cong \Gamma \setminus \Gamma\mathbb{G}(\mathbb{Q}_S)^+ \cong \widetilde{\Gamma} \setminus \mathbb{G}^{sc}(\mathbb{Q}_S),$$

where $\widetilde{\Gamma}$ is the preimage of Γ under a choice of central isogeny $\mathbb{G}^{sc} \to \mathbb{G}$. Now, note that any locally homogeneous space for $\mathbb{G}^{sc}$ is saturated by unipotents by Proposition 3.3.8. Therefore, we can apply the joinings results to these components individually.

3.3.2 Measures on $\mathbb{P}$

In this section, we use the classification of joinings to study measures on $\mathbb{P} = \mathbb{G} \ltimes \mathbb{V}$ which project to the Haar measure on $\mathbb{G}$.

More precisely, let μ be a probability measure on $[\mathbb{P}(\mathbb{A})]$ such that the projection of μ to $[\mathbb{G}(\mathbb{A})]$ is equal to the Haar measure. Suppose further that μ is invariant under A^+. Then it is also invariant under $A' := A^+ \cap \mathrm{im}(a)$, which is a class-$\mathcal{A}'$ subgroup of A^+.

Now, we can decompose μ into its ergodic components (using Proposition 3.1.7) with respect to the class-$\mathcal{A}'$ subgroup A', to get

$$\mu = \int_{\mathcal{M}([\mathbb{P}(\mathbb{A})])^{A'}} \lambda d\mathscr{P}(\lambda),$$

which is supported on the A'-invariant and ergodic measures. Clearly, since $A' \subset A^+ \subset \mathbb{G}(\mathbb{A}_S)^+$, by ergodicity it must be the case that almost all the components λ are supported on a single orbit of $\mathbb{P}(\mathbb{A})^+ := \mathbb{G}(\mathbb{A})^+ \ltimes \mathbb{V}(\mathbb{A})$, say $[\omega\mathbb{P}(\mathbb{A})^+]$. By normality, we get that λ is supported on some $[\mathbb{P}(\mathbb{A})^+\omega]$. In fact, the ergodic components are understood very well as shown in the following Theorem.

Theorem 3.3.9. *Assume $\mathbb{V}$ contains no copy of the trivial representation of $\mathbb{G}$. Let μ be a probability measure on $[\mathbb{P}(\mathbb{A})]$ such that the projection of μ to $[\mathbb{G}(\mathbb{A})]$ is equal to the Haar measure. Suppose μ is A^+-invariant. Then the A^+-ergodic decomposition of μ is of the form*

$$\mu = \int_{\mathcal{M}([\mathbb{P}(\mathbb{A})])^{A'}} \lambda d\mathscr{P}(\lambda)$$

where each λ in the support of $\mathscr{P}$ is the algebraic measure supported on $[\mathbb{L}(\mathbb{A})^+\xi]$ for some $\mathbb{Q}$-group $\mathbb{L} < \mathbb{P}$ which is $\mathbb{G} \ltimes \mathbb{V}'$ for some $\mathbb{G}$-subrepresentation $\mathbb{V}' < \mathbb{V}$.

We will now prove this theorem by applying the joinings theorem successively to S'-adic quotients where $S' \supset S$ is an increasingly large finite set of places. For such a set S', we can consider the quotient

$$X_{\mathcal{K},S'} \cong \mathbb{P}(\mathbb{Q}) \setminus \mathbb{P}(\mathbb{A}_{\mathbb{Q}})/\mathcal{K}^{S'}.$$

Since we have already chosen S large enough such that there is only one S-adic quotient in this adelic quotient, the same will be true for S'. The push-forward measure $\lambda_{S'}$ is A'-invariant and ergodic on $X_{\mathcal{K},S'}$. Furthermore, it will be supported on $[\mathbb{P}(\mathbb{Q}_S)^+\omega]_{S'}$, and its projection to the S'-adic quotient of $\mathbb{G}(\mathbb{Q}_S)$ will be a $\mathbb{G}(\mathbb{Q}_S)^+$-Haar measure.

We are now in the position of Theorem 3.3.6, where we restrict to the locally homogeneous space $[\mathbb{P}(\mathbb{Q}_{S'})^+\omega]$, which is the subject of Theorem 3.3.6 with the $\mathbb{G}_i$'s replaced by their simply connected covers. Thus the measure $\lambda_{S'}$ is an algebraic measure defined over $\mathbb{Q}$. Since this applies to the simply connected cover, we translate that to $X_{\mathcal{K},S'}$ to see that $\lambda_{S'}$ is the periodic measure on $[L_{S'}g_{S'}]$ where $L_{S'}$ is of finite index in $\mathbb{L}(\mathbb{Q}_{S'}) \cap \mathbb{P}(\mathbb{Q}_{S'})^+$ for some $\mathbb{Q}$-subgroup $\mathbb{L} < \mathbb{P}$, which projects onto $\mathbb{G}$ surjectively. At this point, notice that $g_S^{-1}L_S g_S$ contains A', and is a finite index subgroup of an S-adic algebraic group, so it contains A^+. Thus the component

$\lambda_{S'}$ is in fact A^+-invariant and ergodic rather than simply A'-invariant and ergodic. Thus the ergodic decomposition we have given can equally be seen as an A^+-ergodic decomposition.

Lemma 3.3.10. *The group $\mathbb{L}$ is isomorphic to $\mathbb{G} \ltimes \mathbb{V}'$ for some $\mathbb{G}$-subrepresentation $\mathbb{V}' < \mathbb{V}$. In fact, they are conjugate via an element of $\mathbb{V}(\mathbb{Q})$.*

Proof. The first statement is essentially proven in [Kha19b, Theorem 3.2]. We have the short exact sequence,

$$1 \to \mathbb{V} \cap \mathbb{L} \to \mathbb{L} \to \mathbb{G} \to 1.$$

This is the Levi decomposition of $\mathbb{L}$, and so $\mathbb{L} \cong \mathbb{G}' \ltimes \mathbb{V}'$ where $\mathbb{G}' < \mathbb{P}$ surjects onto $\mathbb{G}$ (and therefore is reductive and isogenous to $\mathbb{G}$). Since $\mathbb{G}'$ is a reductive $\mathbb{Q}$-group, it is conjugate over $\mathbb{Q}$ to a subgroup of $\mathbb{G}$ by Lemma 2.5 of *loc. cit.*, and by a simple calculation the conjugating element in $\mathbb{P}(\mathbb{Q})$ can actually be chosen in $\mathbb{V}(\mathbb{Q})$. Therefore we must actually have that $\mathbb{G}'$ is isomorphic (and conjugate) to $\mathbb{G}$. Since conjugation doesn't change a $\mathbb{G}$-subrepresentation of $\mathbb{V}$, we get the result. $\square$

Lemma 3.3.11. *For any $\mathbb{L}$ as described above, $\mathbb{L}(\mathbb{Q}_{S'}) \cap \mathbb{P}(\mathbb{Q}_{S'})^+ = \mathbb{L}(\mathbb{Q}_{S'})^+$. Therefore, $\mathbb{L}(\mathbb{Q}_{S'}) \cap \mathbb{P}(\mathbb{Q}_{S'})^+$ does not contain any proper finite index subgroups.*

Proof. When $\mathbb{L} = \mathbb{G} \ltimes \mathbb{V}'$, recall that $\mathbb{P}(\mathbb{Q}_{S'})^+ = \mathbb{G}(\mathbb{Q}_{S'})^+ \ltimes \mathbb{V}(\mathbb{Q}_S)$, and so the result is clear in this case. When $\mathbb{L} \cong \mathbb{G} \ltimes \mathbb{V}'$ are simply conjugate by an element of $v \in \mathbb{V}(\mathbb{Q})$, then by the fact that unipotence is preserved by conjugation, we see that $\mathbb{L}(\mathbb{Q}_{S'})^+ = v(\mathbb{G}(\mathbb{Q}_{S'})^+ \ltimes \mathbb{V}'(\mathbb{Q}_{S'}))v^{-1}$. By the normality of $\mathbb{P}(\mathbb{Q}_{S'})^+$, we see

$$\mathbb{L}(\mathbb{Q}_{S'})^+ = v(\mathbb{G}(\mathbb{Q}_{S'}) \ltimes \mathbb{V}'(\mathbb{Q}_{S'}))v^{-1} \cap \mathbb{P}(\mathbb{Q}_{S'})^+ = \mathbb{L}(\mathbb{Q}_{S'}) \cap \mathbb{P}(\mathbb{Q}_{S'})^+.$$

The final claim is Corollary 1.5.7 of [Mar91]. $\square$

We now use this S'-adic result to prove the following proposition:

Proof of Theorem 3.3.9. This is a standard procedure, as can be found in the proof of Theorem 4.4 of [Kha17]. We simply need to identify the possible ergodic components λ. By the discussion and lemmas above, we see that the projection, $\lambda_{S'}$ is the periodic measure on $[\mathbb{L}_{S'}(\mathbb{Q}_{S'})^+ g_{S'}]$ for some $g_{S'} \in \mathbb{P}(\mathbb{Q}_{S'})$.

Given an extension $S' \subset S''$, we project $[\mathbb{L}_{S''}(\mathbb{Q}_{S''})^+ g_{S''}]$ to $X_{\mathcal{K},S'}$ and compare it with $[\mathbb{L}_{S'}(\mathbb{Q}_{S'})^+ g_{S'}]$. These must be equal since they are both the projection of λ to $X_{\mathcal{K},S'}$. Therefore,

$$g_{S''}^{-1} \mathbb{L}_{S''}(\mathbb{Q}_{S'})^+ g_{S''} = g_{S'}^{-1} \mathbb{L}_{S'}(\mathbb{Q}_{S'})^+ g_{S'},$$

and $\gamma(g_{S''})_{S'} = l g_{S'} k$ for some $\gamma \in \mathbb{P}(\mathbb{Q})$, $l \in \mathbb{L}_{S'}(\mathbb{Q}_{S'})^+$ and $k \in \mathcal{K}_{S''}^{S'}$. This implies that $\mathbb{L}_{S''} = \mathrm{Ad}_\gamma \mathbb{L}_{S'}$, and in fact after relabelling we can assume they are equal and that $g_{S''} = g_{S'} k$.

Taking a well-ordered direct system of finite sets, $\{S'\}$, each containing S that cover all places of $\mathbb{Q}$, we see that there is a group $\mathbb{L}$, an element $g_S \in \mathbb{P}(\mathbb{Q}_S)$ and $k \in \mathcal{K}^S$ such that every projection is the periodic measure on $[\mathbb{L}(\mathbb{Q}_{S'})^+ g_S k]$. Therefore λ is equal to the periodic measure on $[\mathbb{L}(\mathbb{A})^+ g_S k]$. If $\mathbb{L}$ is conjugate to $\mathbb{G} \ltimes \mathbb{V}'$ by an element $v \in \mathbb{V}(\mathbb{Q})$, we can assume they are equal after replacing $g_S k$ by $v g_S k$. $\square$

The following proposition shows that we can really reduce to the case where V is irreducible. It will be used in the proof of Proposition 2.6.3 at the end of Section 3.6.

Proposition 3.3.12. *For any irreducible $\mathbb{G}$-representation $\mathbb{W}$ and non-zero $\mathbb{Q}$-linear morphism $f : \mathbb{V} \to \mathbb{W}$, we may construct a map*

$$\pi_f : \mathbb{P} \to \mathbb{G} \ltimes \mathbb{W} =: \mathbb{P}_{\mathbb{W}}$$

coming from the identity on $\mathbb{G}$ and the morphism f on $\mathbb{V}$. Suppose, in addition to the assumptions of Theorem 3.3.9, that $\pi_{f,*}\mu$ is a $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$-invariant measure on $[\mathbb{P}_{\mathbb{W}}(\mathbb{A})]$ for any such $f : \mathbb{V} \to \mathbb{W}$, then μ is $\mathbb{P}(\mathbb{A})^+$-invariant on $[\mathbb{P}(\mathbb{A})]$.

Proof. The ergodic decomposition of μ with respect to A^+, when pushed forward by π_f gives an ergodic decomposition

$$\pi_{f,*}\mu = \int \pi_{f,*}\lambda d\mathscr{P}(\lambda).$$

The measure on the left is, by assumption, invariant under $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$, and so we have an A^+-ergodic decomposition of this measure. It suffices to prove that the A^+-ergodic components are also $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$-invariant and ergodic. Assuming this, almost all λ will be such that $\pi_{f,*}\lambda$ is $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$-invariant for all f. This means that the $\mathbb{Q}$-group $\mathbb{G} \ltimes \mathbb{V}'$ associated to λ must surject onto $\mathbb{G} \ltimes W$ for all non-zero maps $f : \mathbb{V} \to \mathbb{W}$. By the semi-simplicity of finite-dimensional representations of semisimple algebraic groups in characteristic 0, [Mil15, Prop 22.137], this implies $\mathbb{V}' = \mathbb{V}$. Therefore almost all A^+-ergodic components of μ are $\mathbb{P}(\mathbb{A})^+$-invariant, so μ is $\mathbb{P}(\mathbb{A})^+$-invariant as required.

It remains to prove the claim (a strengthening of Lemma 3.7 of [Kha19b]) that for a $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$-invariant measure, almost all its A^+-ergodic components are also invariant and ergodic with respect to $\mathbb{P}_W(\mathbb{A})^+$. By lifting to the simply connected cover, we can consider a $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$-invariant measure as a $\mathbb{P}_{\mathbb{W}}^{sc}(\mathbb{A})$-invariant measure on $[\mathbb{P}_{\mathbb{W}}^{sc}(\mathbb{A})]$. The argument of Khayutin using the Mautner phenomena and [GMO08] therefore proves that the A^{sc}-ergodic components of this measure are $\mathbb{P}_{\mathbb{W}}^{sc}(\mathbb{A})$-invariant, and this proves the claim. $\square$

3.3.3 Measures on $\mathbb{G} \times \mathbb{G}$

The proof of the following result is very similar to the proof of the previous section, and in fact is almost identical to Section 4 of [Kha17] so we simply state the result.

Theorem 3.3.13. *Let μ be an $(A^+)^\Delta$-invariant probability measure on $[(\mathbb{G} \times \mathbb{G})(\mathbb{A})]$ and assume that both projections of μ are $\mathbb{G}(\mathbb{A})^+$-invariant. The ergodic decomposition of μ with respect to $(A^+)^\Delta$,*

$$\mu = \int_{\mathcal{M}([(\mathbb{G}\times\mathbb{G})(\mathbb{A})])} \lambda d\mathcal{P}(\lambda), \tag{3.2}$$

has the property that $\mathcal{P}$ is supported on the subset of algebraic measures. Moreover, for almost all the algebraic measures λ in the support of $\mathcal{P}$ the associated $\mathbb{Q}$-group $\mathbb{L} < \mathbb{G} \times \mathbb{G}$ is isogenous either to $\mathbb{G}$ or $\mathbb{G} \times \mathbb{G}$ and projects surjectively on each component, and λ is the algebraic measure supported on $[\mathbb{L}(\mathbb{A})^+\xi]$ for some $\xi \in (\mathbb{G} \times \mathbb{G})(\mathbb{A})$.

3.4 Intermediate Subgroups

It is clear from the Theorems (e.g. 3.3.1 and 3.3.13) that it is important to have a good understanding of the subgroups of maximal rank in algebraic groups over $\mathbb{Q}$. We analyse the general case here, and then investigate specific groups of interest.

Proposition 3.4.1. *Let $\mathbb{G}$ be a reductive group scheme defined over $\mathbb{Q}$. Now, suppose $\mathbb{T}$ is a maximal torus, defined but not necessarily split over $\mathbb{Q}$, such that $\mathbb{T}$ splits over any of the places in S. Let $\mathbb{H}$ be a connected reductive subgroup*

$$\mathbb{T} \leq \mathbb{H} \leq \mathbb{G}$$

defined over $\mathbb{Q}$, necessarily with the same rank as $\mathbb{G}$ over $\mathbb{Q}_S$. Then for any $p \in S$,

$$\mathbb{H}_{\mathbb{Q}_p} = Z_{\mathbb{G}_{\mathbb{Q}_p}} \left(Z \left(\mathbb{H}_{\mathbb{Q}_p} \right) \right)^\circ,$$

i.e. these two groups have the same connected component, and therefore the same Lie algebra.[10] *In fact, the same statement also holds over any finite extension $K/\mathbb{Q}$ for which $\mathbb{T}$ splits.*

[10]Note, the fact that $\mathbb{T}$ splits over S is the crucial condition here that simplifies the set-up compared to the general statement in [Leh12].

This Proposition simply verifies that when the maximal torus is split the results of Borel-de Siebenthal still hold even when the (characteristic 0) field is not algebraically closed. In [Leh12], a general statement is proven over any base scheme. Since we do not require the strength of that result, and the proof of this intermediate result is simpler, we will use this weaker result.

Proof. Pick a place $p \in S$, and consider the groups

$$\mathbb{T}_{\overline{\mathbb{Q}}_p} \leq \mathbb{H}_{\overline{\mathbb{Q}}_p} \leq \mathbb{G}_{\overline{\mathbb{Q}}_p}$$

which are now all split reductive groups over $\overline{\mathbb{Q}}_p$, which is an algebraically closed field of characteristic 0. Consequently, the Lie algebra $\mathfrak{h}_{\overline{\mathbb{Q}}_p} \subset \mathfrak{g}_{\overline{\mathbb{Q}}_p}$ contains the Cartan subalgebra $\mathfrak{t}_{\overline{\mathbb{Q}}_p}$, so is of the form

$$\mathfrak{h}_{\overline{\mathbb{Q}}_p} = \mathfrak{t}_{\overline{\mathbb{Q}}_p} \oplus \bigoplus_{\alpha \in R'} \mathfrak{g}_\alpha$$

for some symmetric (as $\mathcal{H}$ is reductive) subset $R' \subset R$ of the roots of $\mathfrak{g}$. Since $\mathfrak{h}$ is a subalgebra, the subset $R' \subset R$ is closed in the sense that if $\alpha, \beta \in R', \alpha + \beta \in R$ then $\alpha + \beta \in R'$. This follows since we are over an algebraically closed field of characteristic zero by [Mil13, Prop 8.46]. We now wish to show that $\mathbb{Z}R' \cap R = R'$.

Suppose not, then we can find $\rho = \sum_{\alpha \in R'} c_\alpha \alpha \in \mathbb{Z}R' \cap R \setminus R'$, where $c_\alpha \geq 0$ (can be arranged by the symmetry of R'), and $\sum_\alpha c_\alpha$ is minimal in $\mathbb{Z}R' \cap R \setminus R'$.

Now, for each $\beta \in R'$ with $c_\beta > 0$, we must have that

$$\rho - \beta = (c_\beta - 1)\beta + \sum_{\alpha \in R' \setminus \beta} c_\alpha \alpha \notin R'$$

otherwise since R' is closed, we would get $\rho \in R'$, a contradiction. By the minimality of ρ, we must have that $\rho - \beta \notin R$. Therefore, we must have that $\langle \beta, \rho \rangle \leq 0$. By linearity, it follows that $\langle \rho, \rho \rangle \leq 0$, so $\rho = 0$, a contradiction since $0 \notin R$.

Therefore, by [Leh12, Proposition 1.1] (which we may apply by the assumption that $\mathbb{T}$ is split over $\mathbb{Q}_p$), we see that

$$\mathbb{H}_{\mathbb{Q}_p} = Z_{\mathbb{G}_{\mathbb{Q}_p}} \left(Z \left(\mathbb{H}_{\mathbb{Q}_p} \right) \right)^\circ,$$

as required.[11] □

Corollary 3.4.2. *For groups of type A_n^d (i.e. a product of d copies of A_n) over $\mathbb{Q}_S$, the maximal connected reductive subgroups containing a split maximal torus are the product of a Levi subgroup and its centraliser inside the given torus.*

[11] Note that $\mathbb{H}_{\mathbb{Q}_p}$ is connected since $\mathbb{H}$ is, by [Mil15, Prop 1.14].

Proof. By Proposition 3.4.1 and its proof, it suffices to classify the closed root subsystems of the root systems A_n. The results of Borel-de Siebenthal in the algebraically closed case prove that these are given by the root systems of the Levi subgroups. □

3.4.1 Quaternion Algebras

Let B/F be a quaternion algebra over a totally real field F, and consider the group $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}} PB^\times$. The Lie algebra of $\mathbb{G}$ is given by the $\mathbb{Q}$-vector space

$$\mathfrak{g} := B/Z(B).$$

Proposition 3.4.3. *Suppose that $\mathbb{H} \leq \mathbb{G}$ is a connected reductive $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup with maximal rank (i.e. rank $[F : \mathbb{Q}]$) over p. Then $\mathbb{H}$ is either a torus, or is equal to $\mathbb{G}$.*

Proof. Take the preimage, $\widetilde{\mathfrak{h}}$, of $\mathfrak{h} = \mathrm{Lie}(\mathbb{H})$ under the map $B \to \mathfrak{g}$ and extend scalars to $\mathbb{Q}_p$. By Corollary 3.4.2, the corresponding $\mathbb{Q}_p$-vector subspace of B is the Lie algebra of a Levi subgroup (and the entire split torus) which is a semisimple $\mathbb{Q}_p$-algebra (this can be checked simply by looking at the Levi subgroups of GL_n since we have the splitting condition).

Therefore $\widetilde{\mathfrak{h}}$ is a semisimple sub-$\mathbb{Q}$-algebra of B. However, since $\widetilde{\mathfrak{h}}$ has maximal rank, it must contain the centre $F \subset B$, and therefore by Artin-Wedderburn, $\widetilde{\mathfrak{h}}$ is equal to B or K for some quadratic extension K/F. In the first case $\mathbb{H} = \mathbb{G}$, and in the second $\mathbb{H}$ is abelian. □

3.4.2 Unitary Groups

We now use Proposition 3.4.1 to analyse the maximal rank Lie subalgebras of $\mathrm{Lie}(\mathbb{PGU})$, which is a $\mathbb{Q}$-Lie algebra. Now,

$$\mathfrak{g} = \left\{ A = -\overline{A}^T \in M_r(K) \right\}$$

is simply a $\mathbb{Q}$-vector space with the usual Lie bracket.

Proposition 3.4.4. *Suppose that $\mathbb{L} \leq \mathbb{PGU}$ is a reductive $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup with maximal rank over p. Then $\mathbb{L}$ is a unitary group associated to an involution of the second kind*[12] *on $M_n(D)$, where D/K is a division algebra of dimension $m^2[Z(D) : K]$such that $nm[Z(D) : K] = r$. In particular, if r is not prime, there are non-toral proper intermediate subgroups of this kind.*

[12] i.e. one that restricts to complex conjugation on the K

Proof. Let $\mathfrak{h} = \mathrm{Lie}(\mathbb{L}) \leq \mathfrak{g}$. Notice that we no longer have an algebra structure on $\mathfrak{g}$. However, we do have one on $\widetilde{\mathfrak{g}}_K = M_r(K)$ since $\mathbb{PGU}$ splits over K. By the same argument as in Proposition 3.4.3, the Lie algebra $\widetilde{\mathfrak{h}}_K$ will be a semisimple sub-K-algebra of $M_r(K)$, and this algebra is preserved (not pointwise) by the automorphism $A \to \overline{A}^T$. To see this, notice that $\widetilde{\mathfrak{h}}_K$ is the set of elements of the form $h + \zeta l$, where $h, l \in \widetilde{\mathfrak{h}}$ and ζ is any element of $K^\times$ with trace zero to F.

Therefore by Artin-Wedderburn

$$\widetilde{\mathfrak{h}_K} = \prod_{i=1}^{k} M_{n_i}(D_i)$$

where each D_i is a division algebra over K of dimension $m_i^2[Z(D_i) : K]$, and $\sum_i n_i m_i [Z(D_i) : K] = r$. This has k characters defined over $\mathbb{Q}$ (the determinant composed with the norm on each simple factor), and so if $\mathbb{L}$ is $\mathbb{Q}$-anisotropic, we must have $k = 1$ (recalling that we will quotient by the centre). Therefore,

$$\widetilde{\mathfrak{h}}_K \cong M_n(D)$$

and $nm[Z(D) : K] = r$. The involution induced on this is an involution of the second kind (as it restricts to complex conjugation on K), and so the corresponding algebraic group is a unitary group associated to this involution and $M_n(D)$.

For the final statement, suppose that $r = ab$ is not prime, then we can find a degree a totally real extension L/F and then the embedding

$$\mathrm{Res}_{L/F} PGU_b \hookrightarrow PGU_{r,F}$$

gives such an intermediate subgroup. □

This shows that when the rank is not prime, there are serious obstructions to a low entropy method of proving single equidistribution in the unitary case. If r is prime, such a method may work quite well (for a related result see Corollary 1.7 of [Ein+06]).

3.4.3 Intermediate Subgroups in Joint Setting

For the joint setting, the intermediate subgroups to be avoided in Theorem 3.3.13 are much simpler.

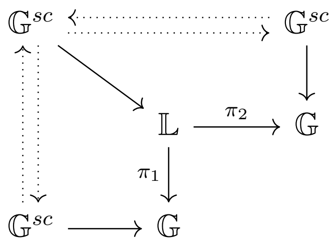
Proposition 3.4.5. *Suppose we fix a splitting* $\mathrm{Out}(G) \to \mathrm{Aut}(G)$ *(just so that we can act on* G *by an outer automorphism). Then in Theorem 3.3.13 we may assume that* $\mathbb{L}$ *is either* $\mathbb{G} \times \mathbb{G}$*, or*

$$\mathbb{G}^{\rho} := \{(x, y) : y = \rho(x)\} < \mathbb{G} \times \mathbb{G}$$

where $\rho = \rho_0 \circ f$ *is the composition of an element* $f \in \mathrm{Aut}_{\mathbb{Q}}(F)$ *is a field automorphism of* $F/\mathbb{Q}$ *and* $\rho_0 \in \mathrm{Out}(G)$ *is an outer automorphism.*

Proof. If $\mathbb{L}$ is isogenous to $\mathbb{G} \times \mathbb{G}$, then $\mathbb{L}(\mathbb{Q}_p) < (\mathbb{G} \times \mathbb{G})(\mathbb{Q}_p)$ has finite index, and therefore $\mathbb{L}(\mathbb{A})^+ = (\mathbb{G} \times \mathbb{G})(\mathbb{A})^+$ (alternatively this follows immediately from Proposition 1.5.5 of [Mar91]).

If $\mathbb{L}$ is isogenous to $\mathbb{G}$, then consider the following diagram



Here, the maps $\mathbb{G}^{sc} \to \mathbb{G}$ and $\mathbb{G}^{sc} \to \mathbb{L}$ are given by the surjective maps with central kernel (the map to $\mathbb{L}$ exists as $\mathbb{L}$ is isogenous to $\mathbb{G}$). The dotted maps exist by the universal property of the simply connected cover. Note that they must all be isomorphisms. Therefore, going from the bottom left to the top right of the diagram constructs an automorphism $\rho \in \mathrm{Aut}(\mathbb{G}^{sc})$ such that $\mathbb{L}$ is the image of the subgroup $\mathbb{G}^{sc} \cong \{(x, \rho(x))\} < \mathbb{G}^{sc} \times \mathbb{G}^{sc}$ under the projection map

$$\mathbb{G}^{sc} \times \mathbb{G}^{sc} \to \mathbb{G} \times \mathbb{G}.$$

The automorphisms of the restriction of scalars of an absolutely almost-simple simply-connected F-group are known by [CGP15, Prop A.5.14]. They are given by a composition of an element of $\mathrm{Aut}(F/\mathbb{Q})$ with an automorphism of G^{sc} (i.e. the group over F). The field automorphisms preserve the inner automorphism group, and so since we may replace $\mathbb{L}$ by any conjugate subgroup in Theorem 3.3.13, we can use just the outer automorphism group of G^{sc}. This is equal to the outer automorphism group of G, and so we obtain the result. □

Since the automorphisms of the field F do not significantly change the problem of bounding the correlation, we will focus on the diagonal possibly twisted by an outer automorphism over F. In fact, in this thesis we will only consider the correlation for the quaternionic groups $G = PB^{\times}$ and so have only the diagonal to discuss.

3.5 Further Results on Entropy

Suppose that we have a measure μ as in Theorem 3.3.1. Then we would like to know an lower bound on its entropy. The measure μ is associated to a reductive linear algebraic subgroup $\mathbb{L} \leq \mathbb{G}$ defined over $\mathbb{Q}$, and in fact the measurable space (X, μ) is measurably isomorphic to a measure on an S-adic quotient of $\mathbb{L}$. Therefore, by the leaf-wise computation of entropy (Theorem 3.2.9), we gain the following proposition:

Proposition 3.5.1. *Let μ be the A-invariant and ergodic probability measure described in Theorem 3.3.1 , and a an element of A. Let $\Phi_{\mathrm{Lyp}}^{\mathbb{L},+} \subset \Phi_{\mathrm{Lyp}}^{+}$ be the subset of positive Lyapunov roots α_ν which are also Lyapunov roots of $\mathbb{L}$. Then, then entropy of μ with respect to a satisfies*

$$h_a(\mu) \leq -\sum_{\alpha_\nu \in \Phi_{\mathrm{Lyp}}^{\mathbb{L},+}} \alpha_\nu(a).$$

Let χ_a denote the maximal value of the right hand side as $\mathbb{L}$ varies over all proper anisotropic subgroups of $\mathbb{G}$. If μ is A-invariant and ergodic on X with $h_a(\mu) > \chi_a$, then μ dominates a non-trivial convex combination of $\mathbb{G}(\mathbb{Q}_S)^+$-invariant probability measures.

This may be useful in proving some surjectivity of the limit of toral measures onto a finite quotient, but is too naive to gain full equidistribution. We will see later that we must consider correlations in order to fully rule out the intermediate measures. We can also consider this in the case of Theorem 3.3.9, where we are considering measures on $\mathbb{P}$ that project to $\mathbb{G}(\mathbb{A})^+$-invariant measures on $[\mathbb{G}(\mathbb{A})]$. In this case we get the following:

Proposition 3.5.2. *Let μ be as in Theorem 3.3.9 with $\mathbb{L} = \mathbb{G}$, where we are assuming now that the representation V is irreducible. Then*

$$h_a(\mu) \leq -\sum_{\alpha_\nu \in \Phi_{\mathrm{Lyp}}^{\mathbb{G},+}} \alpha_\nu(a).$$

Proof. This is clear, since the translate of $\mathbb{G}$ is measurably isomorphic to $\mathbb{G}$ itself. □

We can prove a similar result in the joint setting, except that now it is only the diagonal or twisted diagonal algebraic measures which need to be avoided.

Proposition 3.5.3. *Consider the set-up of Theorem 3.3.13. Suppose that the $\mathcal{P}$-measure of algebraic measures λ supported on $\mathbb{L} \cong \mathbb{G}$ is $0 \leq \epsilon \leq 1$, then*

$$h_a(\mu) = -(2-\epsilon) \sum_{\alpha_\nu \in \Phi_{\text{Lyp}}^{\mathbb{G},+}} \alpha_\nu(a).$$

Another interesting consequence of Proposition 3.4.1 and Theorem 3.2.9 is the fact that in the single equidistribution case we can (up to a point) distinguish algebraic measures on different intermediate subgroups by their entropy. However, since the entropy is only really measuring the action at places S where the torus A^+ has a component, we can only expect to relate the subgroups at the S-adic places. This is precisely what we can do. Note also that we still have no chance of understanding the non-algebraic measures.

Proposition 3.5.4. *Suppose that the two algebraic measures, μ, μ' associated to A^+-invariant homogeneous subsets $[\mathbb{L}(\mathbb{A})^+\xi], [\mathbb{L}'(\mathbb{A})^+\xi'] \subset [\mathbb{G}(\mathbb{A})^+]$ satisfy the property that*

$$\forall a \in A^+, h_\mu(a) = h_{\mu'}(a).$$

Then in fact $\mathbb{L}, \mathbb{L}'$ are forms of the same algebraic group. In particular, for any prime $p \in S$, $\mathbb{L}_{\mathbb{Q}_p}, \mathbb{L}'_{\mathbb{Q}_p}$ are isomorphic split algebraic groups over $\mathbb{Q}_p$.

Proof. The entropy condition translates via the leafwise decomposition of entropy to the statement that the homomorphisms

$$h_\mu(\bullet) = \sum_{\alpha \in \Phi_{\text{Lyp}}^{\mathbb{L},+}} \alpha_\nu\left(\xi \bullet \xi^{-1}\right) : A^+ \to \mathbb{R}$$

and

$$h_{\mu'}(\bullet) = \sum_{\alpha \in \Phi_{\text{Lyp}}^{\mathbb{L}',+}} \alpha_\nu\left(\xi' \bullet (\xi')^{-1}\right) : A^+ \to \mathbb{R}$$

are equal. Since A^+ breaks up as a product of local places, this is equivalent to the local homomorphisms, h_μ^p and $h_{\mu'}^p$, being equal. In the local case, the equality of Proposition 3.4.1

$$\mathbb{L}(\mathbb{Q}_p) = Z_{\mathbb{G}(\mathbb{Q}_p)}\left(Z(\mathbb{L}(\mathbb{Q}_p))^\circ\right.$$

guarantees that h_μ^p vanishes precisely on $Z(\xi^{-1}\mathbb{L}(\mathbb{Q}_p)\xi)$. Therefore

$$Z(\xi^{-1}\mathbb{L}(\mathbb{Q}_p)\xi) = Z\left((\xi')^{-1}\mathbb{L}'(\mathbb{Q}_p)\xi'\right),$$

and again by the above centraliser relation,

$$\xi^{-1}\mathbb{L}(\mathbb{Q}_p)\xi = (\xi')^{-1}\mathbb{L}'(\mathbb{Q}_p)\xi'.$$

Therefore $\mathbb{L}, \mathbb{L}'$ are forms over $\mathbb{Q}$ of the same split group over $\mathbb{Q}_p$. □

This will not be used in this thesis, however we record since it means that were a correlation approach similar to this be applied to a setting with intermediate algebraic measures, the uniformity of the bound in Theorem 3.6.1 need only be proven on a group by group basis.

3.6 Sequences of Toral Sets and Equidistribution

Up until this point, we have considered only a single measure. All of this discussion will be applied to the weak-* limit of a sequence of partial periodic toral measures. We discussed in 3.2.1 the relationship between volumes of Bowen balls and entropy of a single measure. We now adjust this result to consider conditions on the volumes of Bowen balls of the toral measures and how this effects the entropy of the weak-* limit. This will then be compared with the entropy bounds above to deduce equidistribution in general from bounds on the correlations.

Theorem 3.6.1. *Suppose that* $X = [\mathbb{G}(\mathbb{A})]$, $[(\mathbb{G}\times\mathbb{G})(\mathbb{A})]$ *or* $[\mathbb{P}(\mathbb{A})]$, *that* μ_i *is a strict sequence of* A^+*-invariant periodic measures on* X, *and that* μ *is a weak-* limit of* $\{\mu_i\}_i$. *Let* $\mathcal{M}$ *be the space of ergodic* A^+*-invariant measures on* X *so that we can write the ergodic decomposition of* μ *with respect to* A^+ *as*

$$\mu = \int_{\mathcal{M}} \lambda d\mathscr{P}(\lambda),$$

where $\mathscr{P}$ *is a probability measure on* $\mathcal{M}$, *the space of* A^+*-invariant measures on* X. *Let* $a \in A^+$ *be a regular element (meaning that for all Lyapunov roots* α_ν *for places* $\nu \in S$, $\alpha_\nu(a) \neq 0$*). The entropy with respect to* a *is an additive measurable function*

$$h_\bullet(a) : \mathcal{M} \to \mathbb{R}_{\geq 0}.$$

For an interval $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ *and a subset* $C_{\mathcal{M}} \subset \mathcal{M}$, *define* $\mathcal{M}^{\mathcal{I}} := h_\bullet(a)^{-1}(\mathcal{I})$ *and*

$$C_{\mathcal{M}}^{\mathcal{I}} := C_{\mathcal{M}} \cap \mathcal{M}^{\mathcal{I}}.$$

Let $f^n = 1_{B^{(n)}}$ *be the indicator of the Bowen ball* $B^{(n)}$ *of level* $n \geq 1$ *formed using the regular element* $a \in A^+$ *and subset* $B \subset \mathbb{G}(\mathbb{A})$ *or* $\mathbb{P}(\mathbb{A})$. *Suppose that for some finite-length interval* $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ *and compact subset* $C_{\mathcal{M}} \subset \mathcal{M}$, *there are constants* $C, \epsilon > 0$ *(allowed to depend on* $\mathcal{I}, B, C_{\mathcal{M}}$ *but not on* n *and* i*) such that*

$$\forall n \geq 1, \forall \lambda \in C_{\mathcal{M}}^{\mathcal{I}}, \mathrm{Corr}(\mu_i, \lambda)[f^n] \leq Ce^{-2(\sup\mathcal{I}+\epsilon)n} + o_{\mathcal{I},B,C_{\mathcal{M}}}(1) \text{ as } i \to \infty.$$

Then $\mathscr{P}\left(C_{\mathcal{M}}^{\mathcal{I}}\right) = 0$.

Proof. By the strictness assumption, we know that μ is a probability measure. Split the ergodic decomposition up as

$$\mu = \int_{\mathcal{M}^{\mathcal{I}^c}} \lambda d\mathscr{P}(\lambda) + \int_{\mathcal{M}^{\mathcal{I}}} \lambda d\mathscr{P}(\lambda).$$

Since μ is a probability measure, there is a constant c and probability measures μ_0 on X and $\mathscr{P}^{\mathcal{I}}$ on $\mathcal{M}^{\mathcal{I}}$ such that this decomposition is written

$$\mu = (1-c)\mu_0 + c\int_{\mathcal{M}^{\mathcal{I}}} \lambda d\mathscr{P}^{\mathcal{I}}(\lambda).$$

The assumption of the theorem is that $c > 0$ so that we can choose a compact subset, $C_{\mathcal{M}} \subset \mathcal{M}$, such that $\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}}) > 0$, and condition $\mathscr{P}^{\mathcal{I}}$ on this $C_{\mathcal{M}}$ to get a compactly supported probability measure

$$\mathscr{Q}(A) = \frac{\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}} \cap A)}{\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}})}.$$

Since $f = f_B$ is a non-negative test function,

$$\begin{aligned}
\operatorname{Corr}\left(\int \lambda d\mathscr{Q}(\lambda), \int \lambda d\mathscr{Q}(\lambda)\right)[f] &= \frac{1}{c\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}})}\operatorname{Corr}\left(c\int_{C_{\mathcal{M}}^{\mathcal{I}}} \lambda d\mathscr{P}(\lambda), \int \lambda d\mathscr{Q}(\lambda)\right)[f] \\
&\leq \frac{1}{c\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}})}\operatorname{Corr}\left(\mu, \int \lambda d\mathscr{Q}(\lambda)\right)[f] \\
&\leq \frac{1}{c\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}})}\limsup_{i\to\infty}\operatorname{Corr}\left(\mu_i, \int \lambda d\mathscr{Q}(\lambda)\right)[f].
\end{aligned}$$

For contradiction we assume that for some $A, \epsilon > 0$,

$$\forall n \geq 1, \forall \lambda \in \operatorname{supp}(\mathscr{Q}), \operatorname{Corr}(\mu_i, \lambda) \leq Ae^{-2(h+\epsilon)n} + o(1), \text{ as } i \to \infty.$$

where $h = \sup \mathcal{I}$. Since these constants are uniform over $\operatorname{supp}(\mathscr{Q})$, we obtain

$$\operatorname{Corr}\left(\int \lambda d\mathscr{Q}(\lambda), \int \lambda d\mathscr{Q}(\lambda)\right)[f] \leq \frac{A}{c\mathscr{P}^{\mathcal{I}}(C_{\mathcal{M}})}e^{-2(h+\epsilon)n}.$$

Then additivity of entropy and Proposition 3.2.12 implies that

$$\int h_\lambda(a) d\mathscr{Q}(\lambda) = h_{\int \lambda d\mathscr{Q}(\lambda)}(a) > \sup \mathcal{I},$$

which is a clear contradiction since by definition $h_\lambda(a) \leq \sup \mathcal{I}$ for any $\lambda \in h_\bullet(a)^{-1}(\mathcal{I})$. □

This result will ultimately lead us to a proof by contradicting the final inequality. In simpler words, if for a compact collection of ergodic A^+-invariant measures, we can show that the correlation of μ_i with these measures decays rapidly enough, then in fact none of these measures contribute to the limit measure μ. The aim will be to apply this to any compact subset not including the $\mathbb{G}(\mathbb{A})^+$-invariant measures. However, we must point out that the constants A, ϵ that we construct must be uniform over $C_{\mathcal{M}}^{\mathcal{I}}$, so we must pay very close attention to our bounds on the cross correlations to ensure this uniformity. One final piece we must know about toral sets is by how much we can drop the assumption of A^+-invariance. We certainly need splitting at the places in S, but it may be possible to let the particular split torus vary. For this, we must look at the discriminant map from the variety of split tori.

Proposition 3.6.2. *Let p be a finite prime, and $A \subset \mathbb{G}(\mathbb{Q}_p)$ a split maximal torus. The variety of split maximal tori in $\mathbb{G}(\mathbb{Q}_p)$ is given by $\mathbb{G}(\mathbb{Q}_p)/N_{\mathbb{G}(\mathbb{Q}_p)}(A)$, and the discriminant map*

$$\mathrm{disc}_p : \mathbb{G}(\mathbb{Q}_p)/N_{\mathbb{G}(\mathbb{Q}_p)}(A) \to \mathbb{R}^+$$

is proper.

This fact is stated for $\mathbb{G} = \mathrm{SL}_2$ in Khayutin's work, quoting [Ein+06] - here we give the general case.

Proof. The fact that the variety of maximal split tori is equal to $\mathbb{G}(\mathbb{Q}_p)/N_{\mathbb{G}(\mathbb{Q}_p)}(A)$ follows directly from the result of Grothendieck that in $\mathbb{G}(\mathbb{Q}_p)$ all maximal split tori are $\mathbb{G}(\mathbb{Q}_p)$-conjugate.

From the Iwasawa decomposition, we know that if we pick a Borel subgroup containing A, then $\mathbb{G}(\mathbb{Q}_p) = KAN$ for a compact subgroup $K \leq \mathbb{G}(\mathbb{Q}_p)$ and the corresponding unipotent radical N. Consequently, the map

$$K \times N \to \mathbb{G}(\mathbb{Q}_p)/N_{\mathbb{G}(\mathbb{Q}_p)}(A)$$

is surjective, and so to prove the proposition it suffices to prove that the composite map

$$\mathfrak{n} \xrightarrow{exp} N \to \mathbb{G}(\mathbb{Q}_p)/N_{\mathbb{G}(\mathbb{Q}_p)}(A) \to \mathbb{R}^+$$

is proper. By the relation $\mathrm{Ad} \circ \exp = \exp \mathrm{ad}$, we see that for an element $u \in \mathfrak{n}$, the corresponding split torus has a basis of its Lie algebra given by

$$\left\{ \sum_{j=0}^{\infty} \frac{1}{j!} \mathrm{ad}(u)^j f_i \right\}_{i=1,\dots,r}$$

where $\{f_i\}_{i=1,...,r}$ is a basis for the Lie algebra of A. Therefore the discriminant is the denominator of

$$\sum_{j=0}^{\infty} \frac{1}{j!}\mathrm{ad}(u)^j f_1 \otimes ... \otimes \sum_{j=0}^{\infty} \frac{1}{j!}\mathrm{ad}(u)^j f_r.$$

It is easy to see from this that if we express $u = \sum_{\sigma\in\Sigma^+} t_\sigma u_\sigma$ for fixed $u_\sigma \in \mathfrak{n}_\sigma$ and $t_\sigma \in \mathbb{Q}_p$, then the discriminant tends to infinity as $\max_\sigma |t_\sigma|_p \to \infty$, thus proving that this map is proper. □

This proposition means that under the assumption of bounded discriminants we can remove the condition that all the tori are A^+-invariant.

Proposition 3.6.3. *Let $[\mathcal{T}_i l_i] \subset [\mathbb{G}(\mathbb{A})]$ be a sequence of homogeneous toral sets, with associated periodic probability measures μ_i. Provided that*

$$\forall p \in S, \mathrm{disc}_p([\mathcal{T}_i l_i]) \ll 1$$

then there is a pre-compact sequence $\{\xi_i\}_i \subset \mathbb{G}(\mathbb{Q}_S)$ such that $[\mathcal{T}_i l_i \xi_i]$ is A^+-invariant. If the sequence $\{\mu_i\}_i$ is tight and $\mu_i \to \mu$, a probability measure, then by passing to a subsequence we may assume that there is a convergent sequence $\xi_i \to \xi$ such that $\xi_{i,*}\mu_i \to \xi_*\mu$ is a weak- convergence of A^+-invariant probability measures.*

Proof. This follows directly from the fact that the local discriminant maps are proper. □

As would be expected given the analogy between discriminant and torsion, in order to get a suitable A^+-invariant sequence in the Kuga-Sato case we require the local torsion at S to be uniformly bounded as well.

Proposition 3.6.4. *Let $[\mathcal{T}_i(l_i, x_i)] \subset [(\mathbb{G} \ltimes \mathbb{V})(\mathbb{A})]$ be a sequence of K_∞-invariant homogeneous toral sets with associated probability measures μ_i. Provided that*

$$\forall p \in S, \mathrm{disc}_p([\mathcal{T}_i l_i]) \ll 1,$$
$$\forall p \in S, \mathrm{ord}_p([\mathcal{T}_i(l_i, x_i)]) \ll 1,$$

there is a pre-compact sequence $\{\xi_i\}_i \subset \mathbb{P}(\mathbb{A})$ such that $[\mathcal{T}_i(l_i, x_i)\xi_i]$ is A^+-invariant for all i. If the sequence $\{\pi_{\mathbb{G}}\mu_i\}_i$ is tight, then so is $\{\mu_i\}_i$ and if $\mu_i \to \mu$ then by passing to a subsequence we may assume that there is a convergent sequence $\xi_i \to \xi$ such that $\xi_{i,*}\mu_i \to \xi_*\mu$ is a weak- convergence of A^+-invariant probability measures.*

Proof. The boundedness of the order $\mathrm{ord}_{\mathbb{V}(\mathbb{Q}_p)/\mathbb{V}(\mathbb{Z}_p)}(l_p^{-1}x_p)$ means precisely that there is some $m_p \geq 0$ such that for all $i \geq 1$, $l_{i,p}^{-1}x_{i,p} \in p^{-m_p}\mathbb{V}(\mathbb{Z}_p)$. Therefore we can construct $\{\xi_i\}_i$ simply by applying the previous proposition to the first coordinate and the setting the second coordinate of ξ_i equal to $(l_i^{-1}x_i)_p$ for every $p \in S$, and zero otherwise.

If $\{\pi_{\mathbb{G}}\mu_i\}_i$ is tight then since $[\mathbb{P}(\mathbb{A})]$ is a compact extension of $[\mathbb{G}(\mathbb{A})]$ so is the sequence $\{\mu_i\}_i$. □

We can now prove Proposition 2.6.3.

Proof of Proposition 2.6.3. Suppose we have a sequence of $\mathcal{K}_\infty$-invariant homogeneous toral sets on $[\mathbb{P}(\mathbb{A})]$ as is postulated in Conjecture 2.6.2. Since we are assuming that the conjecture is true for $\mathbb{V} = 0$, the sequence $\{\pi_{\mathbb{G}}\mu_i\}_i$ is tight, hence by the previous proposition, we may pass to a subsequence and then have $\xi_i \to \xi$ such that $\xi_{i,*}\mu_i \to \xi_*\mu$ a weak-* convergence of A^+-invariant probability measures on $[\mathbb{P}(\mathbb{A})]$.

Since the conjecture is true for $\mathbb{V} = 0$, we know that the projection of $\xi_*\mu$ to $[\mathbb{G}(\mathbb{A})]$ is $\mathbb{G}(\mathbb{A})^+$-invariant. Also, for any non-zero $\mathbb{G}$-linear map $f : \mathbb{V} \to \mathbb{W}$, the sequence $\{\pi_{f,*}\xi_{i,*}\mu_i\}_i$ on $[(\mathbb{G} \ltimes \mathbb{W})(\mathbb{A})]$ clearly satisfies the conditions for Conjecture 2.6.2 (including strictness) on $\mathbb{G} \ltimes \mathbb{W}$. Thus $\pi_{f,*}\xi_*\mu$ is $\mathbb{P}_{\mathbb{W}}(\mathbb{A})^+$-invariant. Therefore Proposition 3.3.12 applies to show that $\xi_*\mu$ is $\mathbb{P}(\mathbb{A})^+$-invariant, and therefore so is μ as intended. □

We now look more closely at the homogeneous sets in the cases we need, ready for the analysis of correlations later.

Chapter 4

Homogeneous Sets

In this section, we study the details of homogeneous sets for the specific cases that we are interested in.

4.1 Kuga-Sato Case

Let us consider the case of $P = \mathrm{SL}_2 \ltimes \mathbb{G}_a^2$ over a totally real field F. To a homogeneous toral set $[\mathcal{T}\xi]$, where $\xi = (l, x)$, we have an associated torus $T \subset \mathrm{SL}_2$ over F, and this torus must be of the form $\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E}$ for a CM extension E/F. Furthermore, in Section 2.3, we showed how to construct an order in E (by applying the construction to $B = M_2(F)$, with $\mathcal{O} = M_2(\mathcal{O}_F)$).

The F-algebra embedding $E \hookrightarrow M_2(F)$ induces (after picking an arbitrary base point) an identification $\jmath : \mathbb{G}^2_{a,F} \to \mathbb{G}_{a,E}$ such that the subalgebra $E \subset M_2(F)$ acts by multiplication.

In this case, the order in E that we have constructed is the order of the lattice $\mathcal{L} := \bigcap_\nu l_\nu \mathcal{O}^2_{F_\nu} \subset F^2 = E$, since $\Lambda = \bigcap_\nu l_\nu M_2(\mathcal{O}_{F_\nu}) l_\nu^{-1} \cap E_\nu$. Now, contrary to what is stated in [Kha19b], the order Λ may not have trivial class group and so there may be no linear isomorphism $F^2 \to E$ such that $\mathcal{L} \to \Lambda$ is an isomorphism. However, we now have a Λ-ideal $\mathcal{L} \subset E$ which gives us an element in the class group $[\mathfrak{c}] \in \mathrm{Cl}_E(\Lambda)$. The non-existence of the claimed map $\jmath$ of Definition 2.13 in [Kha19b] is not essential, but here we will work with the non-integrally normalised map $\jmath$ defined above. Once we move to the adelic setting, essentially no change is required.

The map $\jmath$ that we have defined above localises to maps $\jmath_\nu : V(F_\nu) \to E_\nu$, and these combine to an adelic map $\jmath_{\mathbb{A}} : V(\mathbb{A}_F) \to \mathbb{A}_E$.

Lemma 4.1.1. *For any place $\nu \in \Sigma_F^\infty$,*

$$\mathrm{Ad}_{l_\nu} \mathrm{SL}_2(\mathcal{O}_F) \cap \mathbb{T}(F_\nu) = \Lambda^{(1)}_\nu.$$

Here $\Lambda_\nu^{(1)}$ refers to the norm 1 elements of Λ_ν, using the norm from Λ_ν to $\mathcal{O}_{F_\nu}$.

Proof. An element of Mat_2 preserves $l_\nu \mathcal{O}_{F_\nu}^2$ if and only if it lies in $\mathrm{Ad}_{l_\nu}\mathrm{GL}_2(\mathcal{O}_{F_\nu})$. Therefore $\mathrm{Ad}_{l_\nu}\mathrm{GL}_2\left(\mathcal{O}_{F_\nu}\right) \cap E_\nu = \Lambda_\nu^\times$ is the invertible elements of the order of $l_\nu \mathcal{O}_{F_\nu}^2 = \mathcal{L}_\nu$. The determinant on $E \subset M_2(F)$ corresponds to the norm map from E to F, and so the result follows. □

There is a corresponding result if we think about the torus inside $P = G \ltimes V$ rather than just inside G. Here the compact object that we consider is $P(\mathcal{O}_{F_\nu}) = \mathrm{SL}_2(\mathcal{O}_{F_\nu}) \ltimes \mathcal{O}_{F_\nu}^2$. The elements of $\mathrm{Ad}_{(l_\nu, x_\nu)} P(\mathcal{O}_{F_\nu})$ are of the form

$$(l_\nu g_\nu l_\nu^{-1}, x_\nu + l_\nu v_\nu - l_\nu g_\nu l_\nu^{-1} x_\nu), g_\nu \in \mathrm{SL}_2(\mathcal{O}_{F_\nu}), v_\nu \in \mathcal{O}_{F_\nu}^2.$$

For such an element to be in $\mathbb{T}(F_\nu) \subset \mathrm{SL}_2(F_\nu) \subset P(F_\nu)$, we require that $v_\nu = g_\nu l_\nu^{-1} x_\nu - l_\nu^{-1} x_\nu \in \mathcal{O}_{F_\nu}^2$. The element $t_\nu = l_\nu g_\nu l_\nu^{-1}$ must not only be in $\Lambda_\nu^{(1)}$, but also satisfy $t_\nu x_\nu - x_\nu \in \mathcal{L}_\nu$. These conditions define a compact open subset

$$\Lambda_\nu^{(1)}(\mathcal{L}, x) := \left\{\lambda \in \Lambda_\nu^{(1)} : \lambda \cdot x_\nu - x_\nu \in \mathcal{L}_\nu\right\} \subset \mathbb{A}_{E,\nu}^{(1)}.$$

We've shown that the volume (as defined in Definition 2.5.1) can be computed as in the following Lemma.

Lemma 4.1.2. *For any place $\nu \in \Sigma_F^\infty$,*

$$\mathrm{Ad}_{(l_\nu, x_\nu)} P(\mathcal{O}_{F_\nu}) \cap \mathbb{T}(F_\nu) = \Lambda_\nu^{(1)}(\mathcal{L}, x).$$

Therefore,

$$\mathrm{vol}([\mathcal{T}\xi) = [\Lambda_f^{(1)}(\mathcal{L}, x) : \mathcal{T} \cap \Lambda_f^{(1)}(\mathcal{L}, x)][T(\mathbb{A}_F) : \mathcal{T}] m_{\mathbb{T}(\mathbb{A}_f)} \left(\Lambda_f^{(1)}(\mathcal{L}, x)\right)^{-1}.$$

We define

$$f_{\mathcal{L},x}(\mathcal{T}) := \frac{\left[\Lambda_f^{(1)}(\mathcal{L}, x) : \mathcal{T} \cap \Lambda_f^{(1)}(\mathcal{L}, x)\right]}{[\mathbb{T}(\mathbb{A}_F) : \mathcal{T}]}$$

*to be the **inertia** of the subset $\mathcal{T} \subset T(\mathbb{A}_F)$.*

The reason for the terminology introduced above is that if $\mathcal{T}$ is the norm subgroup attached to an extension of E, then this quantity is closely related to the inertial index of the extension.

To get a better formula for the volume above, we must first prove some simple properties of orders for CM extensions.

Lemma 4.1.3. *Let $\Lambda_\nu \subset \mathcal{O}_{E_\nu}$ be a local $\mathcal{O}_{F_\nu}$-order in the ν-adic integers of a quadratic etale extension E_ν/F_ν. Then if $\mathfrak{p}_\nu \subset \mathcal{O}_{F_\nu}$ is the maximal order, and $n \geq 1$ is minimal such that $\mathfrak{p}^n\mathcal{O}_{E_\nu} \subset \Lambda_\nu$ (in particular $\Lambda_\nu \neq \mathcal{O}_{E_\nu}$), then*

$$\Lambda_\nu = \mathcal{O}_{F_\nu} + \mathfrak{p}^n\mathcal{O}_{E_\nu}.$$

Also, we get the following volumes:

$$\begin{aligned} m(\mathcal{O}_{E_\nu}^\times) &= \begin{cases} N\mathfrak{p}^{-2}\left(1 - N\mathfrak{p}^{-1}\right)^2, & \text{if } E_\nu/F_\nu \text{ is split,} \\ N\mathfrak{p}^{-2}\left(1 - N\mathfrak{p}^{-2}\right), & \text{if } E_\nu/F_\nu \text{ is non-split} \end{cases} \\ &= L_{E_\nu/F_\nu}(1)^{-1}. \\ m(\Lambda_\nu^\times) &= N\mathfrak{p}^{-n}(1 - N\mathfrak{p}^{-1}) \\ m(\Lambda_\nu) &= N\mathfrak{p}^{-n}. \end{aligned}$$

Therefore the index of the units in a non-maximal order is

$$\left[\mathcal{O}_{E_\nu}^\times : \Lambda_\nu^\times\right] = N\mathfrak{p}^n L_\nu(\chi_{E_\nu/F_\nu}, 1)^{-1}.$$

Proof. Suppose that $\Lambda_\nu \neq \mathcal{O}_{F_\nu} + \mathfrak{p}_\nu^n\mathcal{O}_{E_\nu}$, and so we can find an element $x + \pi_\nu^{n-1}y \in \Lambda_\nu$ such that $x \in \mathcal{O}_{F_\nu}$ and $y \in \mathcal{O}_{E_\nu} \setminus (\mathcal{O}_{F_\nu} + \mathfrak{p}_\nu\mathcal{O}_{E_\nu})$. Then $\pi_\nu, \pi_\nu^{n-1}y \in \Lambda_\nu$ and the $\mathcal{O}_{F_\nu}$-span of these elements is $\pi^{n-1}\mathcal{O}_{E_\nu}$, by the condition that $y \notin \mathcal{O}_{F_\nu} + \mathfrak{p}_\nu\mathcal{O}_{E_\nu}$. Therefore $\mathcal{O}_{F_\nu} + \pi^{n-1}\mathcal{O}_{E_\nu} \in \Lambda_\nu$, which is a contradiction.

The volumes are simple to calculate. For example, it is immediate to see that

$$m(\Lambda_\nu^\times) = N\mathfrak{p}^{-2}\left|\mathcal{O}_{F_\nu}^\times/(1 + \mathfrak{p}^n\mathcal{O}_{F_\nu})\right|,$$

which gives the desired result. The index calculation then comes from the ratio $m(\mathcal{O}_{E_\nu}^\times/\Lambda_\nu^\times)$. □

We now give a more detailed calculation of the volume of the tori in the Kuga-Sato case. The reason that we wish to compute the volume in terms the L-value is that this will appear via the sieve theory analysis of the correlations and these two L-values will be equated.

Proposition 4.1.4. *The volume is*

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = \frac{2}{(2\pi)^{[F:\mathbb{Q}]}}\frac{L(\chi_{E/F}, 1)}{2^{\#\mathrm{Ram}_F(E)}}\sqrt{\left|\frac{D_E}{D_F}\right|}\left|\widehat{\mathcal{O}}_E^{(1)} : \Lambda_f(\mathcal{L}, x)^{(1)}\right| f_{\mathcal{L},x}(\mathcal{T}).$$

Proof. We know that

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = f_{\mathcal{L},x}(\mathcal{T}) m_{\mathbb{T}(\mathbb{A}_f)} \left(\Lambda_f^{(1)}(\mathcal{L},x)\right)^{-1}.$$

We leave the inertia alone, and compute the volume term as in [Kha19b]. Since we are assuming the normalisation of the measure such that $\mathbb{T}(\mathbb{Q}) \setminus \mathbb{T}(\mathbb{A})$ has volume 1, we see that the volume term is

$$m_{\mathbb{T}(\mathbb{A}_f)} \left(\Lambda_f^{(1)}(\mathcal{L},x)\right) = \left|\mathbb{T}(\mathbb{Q}) \setminus \mathbb{T}(\mathbb{A}_f)/\Lambda_f^{(1)}(\mathcal{L},x)\right|^{-1} \cdot \left|\Lambda_f^{(1)}(\mathcal{L},x) \cap \mathbb{T}(\mathbb{Q})\right|.$$

The simple term is

$$\left|\Lambda_f^{(1)}(\mathcal{L},x) \cap \mathbb{T}(\mathbb{Q})\right| = \mu_{\Lambda(\mathcal{L},x)} := \mu_E \cap \Lambda(\mathcal{L},x)$$

is the set of roots of unity which lie in $\Lambda(\mathcal{L},x)$. There is a map

$$\imath_{pg} : \mathbb{T}(\mathbb{Q}) \setminus \mathbb{T}(\mathbb{A}_f)/\Lambda_f^{(1)}(\mathcal{L},x) \to E^\times \setminus \mathbb{A}_{E,f}^\times/\Lambda_f(\mathcal{L},x)^\times$$

which may not be injective. The kernel is $\left(E^\times \Lambda_f(\mathcal{L},x)\right)^{(1)}/E^{(1)}\Lambda_f^{(1)}(\mathcal{L},x)$. By taking norms, we get an isomorphism

$$\ker \imath_{pg} \cong NE^\times \cap N\Lambda_f(\mathcal{L},x)^\times/N(E^\times \cap \Lambda_f(\mathcal{L},x)^\times).$$

In the case of $F = \mathbb{Q}$, this quotient is trivial since $\mathbb{Q}^\times \cap \widehat{\mathbb{Z}}^\times \mathbb{R}_{>0} = \{1\}$. However in the general case it may not be trivial.

We can compute the index $\left[E^\times \setminus \mathbb{A}_{E,f}^\times/\Lambda_f(\mathcal{L},x)^\times : \mathrm{im}\,\imath_{pg}\right]$ using the following exact sequence:

$$1 \to \mathrm{im}\,\imath_{pg} \to E^\times \setminus \mathbb{A}_{E,f}^\times/\Lambda_f(\mathcal{L},x)^\times \xrightarrow{\mathrm{Nr}} F^\times \setminus \mathbb{A}_F^\times/F_\infty^{\gg 0}\mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times \to \{\pm 1\} \to 1.$$

Therefore the index is equal to

$$\frac{1}{2}\left|F^\times \setminus \mathbb{A}_F^\times/F_\infty^{\gg 0}\mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times\right|.$$

We can relate this to the narrow class group of F by the snake lemma for the following diagram:

This tells us that

$$\left|F^\times \setminus \mathbb{A}_F^\times/F_\infty^{\gg 0}\mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times\right| = \frac{\left|\widehat{\mathcal{O}}_F^\times : \mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times\right|}{\left|\mathcal{O}_F^{\times,\gg 0} : \mathcal{O}_F^{\times,\gg 0} \cap \mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times\right|} h_F^+$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \frac{F_\infty^{\gg 0}\mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times}{F_\infty^{\gg 0}\mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times \cap F^\times} & \longrightarrow & F^\times \setminus \mathbb{A}_F^\times & \longrightarrow & F^\times \setminus \mathbb{A}_F^\times / F_\infty^{\gg 0}\mathrm{Nr}\Lambda_f(\Lambda,x)^\times & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \frac{F_\infty^{\gg 0}\widehat{\mathcal{O}}_F^\times}{\mathcal{O}_F^{\times,\gg 0}} & \longrightarrow & F^\times \setminus \mathbb{A}_F^\times & \longrightarrow & \mathrm{Cl}_F^+ & \longrightarrow & 1
\end{array}$$

The same computation tells us that

$$\left|E^\times \setminus \mathbb{A}_{E,f}^\times / \Lambda_f(\mathcal{L},x)^\times\right| = \frac{\left|\widehat{\mathcal{O}}_E^\times : \Lambda_f(\mathcal{L},x)^\times\right|}{\left|\mathcal{O}_E^\times : \mathcal{O}_E^\times \cap \Lambda_f(\mathcal{L},x)^\times\right|} h_E.$$

Putting these together we compute that

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = \frac{2|\mathcal{O}_F^{\times,\gg 0} : \mathrm{Nr}\left(E^\times \cap \Lambda_f(\mathcal{L},x)^\times\right)||\widehat{\mathcal{O}}_E^\times : \Lambda_f(\mathcal{L},x)^\times| h_E f_{\mathcal{L},x}(\mathcal{T})}{\mu_{\Lambda(\mathcal{L},x)}\left|\widehat{\mathcal{O}}_F^\times : \mathrm{Nr}\Lambda_f(\mathcal{L},x)^\times\right|\left|\mathcal{O}_E^\times : \mathcal{O}_E^\times \cap \Lambda_f^\times(\mathcal{L},x)\right| h_F^+}.$$

We can simplify this by considering the snake lemma for the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^{(1)} \cap \Lambda_f^{(1)}(\mathcal{L},x) & \longrightarrow & E^\times \cap \Lambda_f^\times(\mathcal{L},x) & \longrightarrow & \mathrm{Nr}(E^\times \cap \Lambda_f(\mathcal{L},x)^\times) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & E^{(1)} \cap \widehat{\mathcal{O}}_E^\times & \longrightarrow & \mathcal{O}_E^\times & \longrightarrow & \mathrm{Nr}(\mathcal{O}_E^\times) & \longrightarrow & 1
\end{array}$$

This tells us that

$$\frac{\mu_E}{\mu_{\Lambda(\mathcal{L},x)}}|\mathrm{Nr}\mathcal{O}_E^\times : \mathrm{Nr}\left(E^\times \cap \Lambda_f(\mathcal{L},x)^\times\right)| = |\mathcal{O}_E^\times : \mathcal{O}_E^\times \cap \Lambda_f^\times(\mathcal{L},x)|,$$

and therefore

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = \frac{2}{h_F^+}\frac{\left|\mathcal{O}_F^{\times,\gg 0} : \mathrm{Nr}\mathcal{O}_E^\times\right| h_E}{\mu_E}\frac{\left|\widehat{\mathcal{O}}_E^\times : \Lambda_f(\mathcal{L},x)^\times\right|}{\left|\widehat{\mathcal{O}}_F^\times : \mathrm{Nr}\Lambda_f^\times(\mathcal{L},x)\right|} f_{\mathcal{L},x}(\mathcal{T}).$$

We have arranged this into 4 terms: the first depends only on the base field F, the second are global terms which only additionally depend on the embedded field E, the third are local terms which additionally depend on the order of the embedding, and the fourth additionally depends on the subtorus $\mathcal{T}$. We can now apply the class number formula to the terms h_F^+ and h_E. To do this, we note that

$$h_F^+ = h_F 2^{[F:\mathbb{Q}]}\left|\mathcal{O}_F^\times : \mathcal{O}_F^{\times,\gg 0}\right|^{-1},$$

and if we set $Q = \left[\mathcal{O}_E^\times : \mu_E \mathcal{O}_F^\times\right] = \left[N\mathcal{O}_E^\times : \left(\mathcal{O}_F^\times\right)^2\right] = 1$ or 2, then $R_F = 2^{1-d} Q R_E$, and the analytic class number formula says that

$$\frac{2h_E}{\mu_E h_F^+} = \frac{\left|\mathcal{O}_F^\times : \mathcal{O}_F^{\times,\gg 0}\right|}{(2\pi)^{[F:\mathbb{Q}]}} 2^{1-[F:\mathbb{Q}]} Q \sqrt{\left|\frac{D_E}{D_F}\right|} L(\chi_{E/F}, 1).$$

Now we can use the fact that

$$\left|\mathcal{O}_F^\times : \mathrm{Nr}\mathcal{O}_E^\times\right| = 2^{[F:\mathbb{Q}]} Q^{-1}$$

to give

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = \frac{2}{(2\pi)^{[F:\mathbb{Q}]}} L(\chi_{E/F}, 1) \sqrt{\left|\frac{D_E}{D_F}\right|} \frac{\left|\widehat{\mathcal{O}}_E^\times : \Lambda_f(\mathcal{L}, x)^\times\right|}{\left|\widehat{\mathcal{O}}_F^\times : \mathrm{Nr}\Lambda_f^\times(\mathcal{L}, x)\right|} f_{\mathcal{L},x}(\mathcal{T}).$$

The final step is to use the snake lemma for the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda_f^{(1)}(\mathcal{L}, x) & \longrightarrow & \Lambda_f^\times(\mathcal{L}, x) & \longrightarrow & \mathrm{Nr}\Lambda_f^\times(\mathcal{L}, x) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \widehat{\mathcal{O}}_E^{(1)} & \longrightarrow & \widehat{\mathcal{O}}_E^\times & \longrightarrow & \mathrm{Nr}\widehat{\mathcal{O}}_E^\times & \longrightarrow & 1 \end{array}$$

which tells us that

$$\frac{\left|\widehat{\mathcal{O}}_E^\times : \Lambda_f(\mathcal{L}, x)^\times\right|}{\left|\widehat{\mathcal{O}}_F^\times : \mathrm{Nr}\Lambda_f^\times(\mathcal{L}, x)\right|} = \frac{\left|\widehat{\mathcal{O}}_E^{(1)} : \Lambda_f^{(1)}(\mathcal{L}, x)\right|}{2^{\#\mathrm{Ram}_F(E)}}$$

□

We can actually give a more general proof of the volume formula applicable in all cases using the work of [Shy77], however this still leaves us with a difficult discriminant computation.

Proposition 4.1.5. *In the general case of a homogeneous toral set* $[\mathcal{T}\xi]$ *arising from an embedding* $\mathbb{T} \to \mathbb{G}$ *we get a compact open subgroup* $\mathcal{K}_T \subset \mathbb{T}(\mathbb{A}_f)$ *from* $\mathrm{Ad}_\xi \mathbb{P}(\widehat{\mathbb{Z}}) \cap \mathbb{T}(\mathbb{A}_f)$ *or* $\mathrm{Ad}_\xi(\mathbb{G} \times \mathbb{G})(\widehat{\mathbb{Z}}) \cap \mathbb{T}(\mathbb{A}_f)$. *Define the* ***inertia*** *as before to be*

$$f_{\mathcal{K}_T}(\mathcal{T}) := \frac{[\mathcal{K}_T : \mathcal{T} \cap \mathcal{K}_T]}{[\mathbb{T}(\mathbb{A}_f) : \mathcal{T}]},$$

and the ***embedding index*** *to be*

$$e_{\mathcal{K}_T} := [\mathcal{K}_T^{max} : \mathcal{K}_T]$$

where $\mathcal{K}_T^{max}$ is the maximal open compact subgroup of $\mathbb{T}(\mathbb{A}_f)$. Then

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = \tau_{\mathbb{T}} L_{\mathbb{T}}(1) D_{\mathbb{T}}^{1/2} e_{\mathcal{K}_T} f_{\mathcal{K}_T}(\mathcal{T}),$$

where $\tau_{\mathbb{T}}, L_{\mathbb{T}}, D_{\mathbb{T}}$ are the Tamagawa number, Artin L-function and discriminant attached to $\mathbb{T}$ by Ono ([Ono61]).

Proof. As before it is clear that

$$\mathrm{vol}\left([\mathcal{T}\xi]\right) = m_{\mathbb{T}(\mathbb{A}_f)}\left(\mathcal{K}_T\right)^{-1} f_{\mathcal{K}_T}(\mathcal{T})$$

and therefore we simply need to compute the volume. This time, we note that

$$\begin{aligned} m_{\mathbb{T}(\mathbb{A}_f)}(\mathcal{K}_T)^{-1} &= \left[\mathcal{K}_T^{max} : \mathcal{K}_T\right] m_{\mathbb{T}(\mathbb{A}_f)}(\mathcal{K}_T^{max})^{-1} \\ &= e_\Lambda \frac{h_{\mathbb{T}}}{w_{\mathbb{T}}} \end{aligned}$$

using again the notation of [Ono61]. Now by the class number formula for tori of [Shy77], we get the desired result, using that for anisotropic tori $R_{\mathbb{T}} = 1$ and $L_{\mathbb{T}}$ has no pole at $s = 1$. □

Why does this imply the previous result? It follows from the computations of [Shy77] and the short exact sequence of tori

$$1 \to \mathbb{T} \to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to 1$$

that the discriminant $D_{\mathbb{T}}^{1/2}$ is equal to

$$D_{\mathbb{T}}^{1/2} = \left(\frac{D_{\mathbb{G}_{m,E}}}{D_{\mathbb{G}_{m,F}}}\right)^{1/2} \frac{q((\widehat{\lambda})_\infty)}{q(\lambda_\infty)} \prod_p q(\lambda_p^c)^{-1},$$

where $\lambda : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \to \mathbb{T} \times \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the isogeny $x \mapsto \left(\frac{x^2}{\mathrm{Nm}(x)}, \mathrm{Nm}(x)\right)$, and the induced maps are

$$\left(\widehat{\lambda}\right)_\infty : \left(\widehat{\mathbb{T}}\right)^\infty \times \left(\widehat{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m}\right)^\infty \to \left(\widehat{\mathrm{Res}_{E/\mathbb{Q}}}\right)^\infty$$

between the $\mathrm{Gal}\left(E/F\right)$-fixed points of the character lattices, and

$$\begin{aligned} \lambda_\infty &: E_\infty^\times \to \mathbb{T}(\mathbb{R}) \times F_\infty^\times, \\ \lambda_p^c &: \mathcal{O}_{E_p}^\times \to \mathbb{T}(\mathbb{Z}_p) \times \mathcal{O}_{F_p}^\times, \end{aligned}$$

and the q-symbol for a map is given by $q(f) = |\mathrm{coker}(f)| \,/\, |\ker(f)|$ if this exists. The discriminants for the multiplicative group for any number field L are known to be

$$D_{\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m} = \frac{1}{2^{2r_1(L)}(2\pi)^{2r_2(L)}}|D_L|.$$

Now the calculations for the q-symbols are as follows:

$$\begin{aligned}
&\ker\left(\left(\widehat{\lambda}\right)_\infty\right)=0, && \operatorname{coker}\left(\left(\widehat{\lambda}\right)_\infty\right)=0, && q\left(\left(\widehat{\lambda}\right)_\infty\right)=1,\\
&\ker\left(\lambda_\infty\right)=\{\pm1\}^{[F:\mathbb{Q}]}, && \operatorname{coker}\left(\lambda_\infty\right)=\left(\mathbb{R}^\times/\mathbb{R}_+^\times\right)^{[F:\mathbb{Q}]}, && q(\lambda_\infty)=1,\\
&\ker\left(\lambda_2^c\right)=\{\pm1\}^{\left|\Sigma_{F,2}\right|}, && \operatorname{coker}\left(\lambda_2^c\right)=(\mathbb{Z}/2)^{[F:\mathbb{Q}]+\left|\Sigma_{F,2}\right|+|\mathrm{Ram}_2(E/F)|}, && q(\lambda_2^c)=2^{[F:\mathbb{Q}]+|\mathrm{Ram}_2(E/F)|},\\
&\ker\left(\lambda_p^c\right)=\{\pm1\}^{\left|\Sigma_{F,p}\right|}, && \operatorname{coker}\left(\lambda_p^c\right)=(\mathbb{Z}/2)^{\left|\Sigma_{F,p}\right|+|\mathrm{Ram}_p(E/F)|}, && q(\lambda_p^c)=2^{|\mathrm{Ram}_p(E/F)|},(p\neq 2).
\end{aligned}$$

Finally, $\tau_{\mathbb{T}}=2$ as a special orthogonal group, and so we recover the previous proposition. To see this another way, note that by Proposition 4.5.1 of [Ono63], $\tau_T=\#H^1(\mathbb{Z}/2,\mathbb{Z})=2$ where the action of $\mathbb{Z}/2$ on $\mathbb{Z}$ is the non-trivial one (and $\tau_T=\tau_{\mathbb{T}}$ by Section 3 of *loc. cit.*).

4.1.1 Larger Homogeneous Sets

In the Kuga-Sato case we are also interested in homogeneous sets of the form $[(\mathbb{G}\ltimes\mathbb{V}')(\mathbb{A})(g,y)]$. We call these homogeneous Hecke sets. For these sets, the volume is similarly defined:

Definition 4.1.6.

$$\operatorname{vol}\left([(\mathbb{G}\ltimes\mathbb{V}')(\mathbb{A})(g,y)]\right)=m_{\mathbb{G}\ltimes\mathbb{V}'}\left((g,y)\left(B_{\mathbb{G}}\ltimes B_{\mathbb{V}}\right)(g,y)^{-1}\right)^{-1}.$$

Since $\mathrm{Ad}_{(g,0)}$ will preserve $(\mathbb{G}\ltimes\mathbb{V}')$ we can replace (g,y) by $(e,g^{-1}y)$ in the definition, and so we find

$$\operatorname{vol}\left([(\mathbb{G}\ltimes\mathbb{V}')(\mathbb{A})(g,y)]\right)=\int_{B_{\mathbb{G}}} m_{\mathbb{V}'}\left(\mathbb{V}(\widehat{\mathbb{Z}})+g^{-1}y-h\left(g^{-1}y\right)\right)dh.$$

For a fixed h, if there is an element $\alpha\in\mathbb{V}(\widehat{\mathbb{Z}})$ such that $\alpha+g^{-1}y-h(g^{-1}y)\in\mathbb{V}'(\mathbb{A})$, then

$$\left(\mathbb{V}\left(\widehat{\mathbb{Z}}\right)+g^{-1}y-h\left(g^{-1}y\right)\right)\cap\mathbb{V}'(\mathbb{A})=\alpha+g^{-1}y-h\left(g^{-1}y\right)+\mathbb{V}'\left(\widehat{\mathbb{Z}}\right)$$

which has volume 1 as it is simply a translation of $\mathbb{V}'\left(\widehat{\mathbb{Z}}\right)$. If such an $\alpha\in\mathbb{V}\left(\widehat{\mathbb{Z}}\right)$ does not exist then clearly the volume is zero. Therefore, if $\pi:\mathbb{V}\to\mathbb{W}$ has kernel $\mathbb{V}'$ then the volume of this intermediate homogeneous set is equal to the index

$$\operatorname{vol}\left([(\mathbb{G}\ltimes\mathbb{V}')(\mathbb{A})(g,y)]\right)=\left[\mathbb{G}\left(\widehat{\mathbb{Z}}\right):\mathrm{Stab}_{\mathbb{G}(\widehat{\mathbb{Z}})}\left(\pi\left(g^{-1}y\right)+\pi\mathbb{V}\left(\widehat{\mathbb{Z}}\right)\right)\right].$$

A clear necessary condition for equidistribution of a sequence of homogeneous torus orbits is that we require the volume of intermediate homogeneous sets which contain the torus sets must tend to infinity.

Proposition 4.1.7. *The intermediate homogeneous sets containing* $[\mathbb{T}(\mathbb{A})(l,x)]$ *all have the form*

$$[(\mathbb{G} \ltimes \mathbb{V}')(\mathbb{A})(l,x)], \text{ for } \mathbb{V}' \leq \mathbb{V}.$$

Proof. Clearly we can assume the intermediate homogeneous set has the form $[(\mathbb{G} \ltimes \mathbb{V}')(\mathbb{A})(l, x+w$ where $w \in \mathbb{W}(\mathbb{A})$ for a chosen complement $\mathbb{W} \leq \mathbb{V}$ defined over $\mathbb{Q}$ such that $\mathbb{V} = \mathbb{V}' \oplus \mathbb{W}$. The inclusion holds if $\forall t \in \mathbb{T}(\mathbb{A}), \exists \gamma \in \mathbb{G}(\mathbb{Q}), v \in \mathbb{V}(\mathbb{Q}), g \in \mathbb{G}(\mathbb{A}), v' \in \mathbb{V}'(\mathbb{A})$ such that

$$tl = \gamma g l, tx = v + \gamma v' + \gamma g(x+w).$$

Thus, we require $v \in \mathbb{V}(\mathbb{Q})$ such that

$$v + tw \in \mathbb{V}'(\mathbb{A}).$$

Since we gave a decomposition $\mathbb{V} = \mathbb{V}' \oplus \mathbb{W}$ over $\mathbb{Q}$, the above is possible if and only if $tw \in \mathbb{W}(\mathbb{Q})$ for all $t \in \mathbb{T}(\mathbb{A})$. This is true only if $w \in \mathbb{W}^{\mathbb{T}}(\mathbb{Q})$ is fixed by the action of $\mathbb{T}$.

Suppose that $w \neq 0$. Then $\mathbb{T} \leq \mathbb{H} := \mathrm{Stab}_{\mathbb{G}}(w)$ and since we have assumed that $\mathbb{V}$ contains no copies of the trivial representation of $\mathbb{G}$, we conclude that $\mathbb{H}$ is a proper parabolic subgroup containing an anisotropic maximal torus, a contradiction. Thus $w = 0$.

A simple calculation shows that there is an inclusion of intermediate homogeneous sets

$$[(\mathbb{G} \ltimes \mathbb{V}')(\mathbb{A})(l,x)] \subset [(\mathbb{G} \ltimes \mathbb{V}'')(\mathbb{A})(l,x)]$$

if and only if $\mathbb{V}' \leq \mathbb{V}''$. □

This Proposition, together with the computation of the volume of an intermediate homogeneous set tells us that a necessary condition for equidistribution is that

$$\min_{\pi:\mathbb{V}\to\mathbb{W}} \left[\mathbb{G}(\widehat{\mathbb{Z}}) : \mathrm{Stab}_{\mathbb{G}(\widehat{\mathbb{Z}})}\left(\pi(l_i^{-1}x_i) + \pi\mathbb{V}(\widehat{\mathbb{Z}})\right)\right] \to \infty.$$

This constitutes part of the definition of strictness of the sequence of torus orbits. Note that practically, Proposition 2.6.3 shows that in this section we need only consider $\mathbb{V}' = 0$ or $\mathbb{V}$.

4.2 Homogeneous Sets in the Joint Case

The computation of the volume $\mathrm{vol}([\mathcal{T}(g, sg)])$ in the joint case is similar to the volume computation in the Kuga-Sato case. By Proposition 4.1.5, we get that

$$\mathrm{vol}([\mathcal{T}(g, sg)]) = 2f_{\mathcal{K}_T}(\mathcal{T})e_{\mathcal{K}_T}L(1, \chi_{E/F})D_{\mathbb{T}}^{1/2}$$

where $\mathcal{K}_T = \mathrm{Ad}_g\mathcal{K}_f \cap T(\mathbb{A}_F^\infty)$ is the compact subgroup corresponding to the order $\Lambda \subset E$. Note however that in this situation, the maximal compact subgroup of $T(\mathbb{A}_{F,f})$ is not the image of $\mathcal{O}_{E,f}^\times$. In fact it is larger than this by a factor of $2^{\mathrm{Ram}_{E/F}}$. We now compute the q-numbers as before, but now for the short exact sequence

$$1 \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \to \mathbb{T} \to 1.$$

This induces the map $\lambda : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \to \mathbb{T} \times \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ given by $x \mapsto (x, \mathrm{Nm}(x))$. We see that in fact the q-numbers for this map are identical to those in the Kuga-Sato case. Therefore,

$$\mathrm{vol}([\mathcal{T}(g, sg)]) = 2f_{\mathcal{K}_T}(\mathcal{T})e_{\mathcal{K}_T}L(1, \chi_{E/F})\sqrt{\left|\frac{D_E}{D_F}\right|}(2\pi)^{-[F:\mathbb{Q}]}2^{-\mathrm{Ram}_{E/F}}$$

Now, by Lemma 4.1.3, we can see that

$$2^{-\mathrm{Ram}_{E/F}}e_{\mathcal{K}_T} = \left[\mathcal{O}_{E,f}^\times : \Lambda_f^\times\right] = (N\mathfrak{f})L_{\mathfrak{f}}(1, \chi_{E/F})^{-1}$$

where $L_{\mathfrak{f}}(1, \chi_{E/F}) = \prod_{\nu|\mathfrak{f}} L_\nu(1, \chi_{E/F})$. We summarise this below:

Lemma 4.2.1. *The volume of a joint homogeneous toral set* $[\mathcal{T}(g, sg)]$ *is given by*

$$\mathrm{vol}([\mathcal{T}(g, sg)]) = \frac{2(N\mathfrak{f})[\mathcal{K}_T : \mathcal{T} \cap \mathcal{K}_T]}{(2\pi)^{[F:\mathbb{Q}]}[T(\mathbb{A}_F) : \mathcal{T}]}L^{\mathfrak{f}}(1, \chi_{E/F})\sqrt{\left|\frac{D_E}{D_F}\right|}$$

where $\mathfrak{f} \subset \mathcal{O}_F$ *is the conductor of the order* Λ *attached to* $[\mathcal{T}g]$*, and* $\mathcal{K}_T = T(\mathbb{A}_{F,f}) \cap g\mathcal{K}_f g^{-1}$.

Recall that for the main result Theorem 2.6.5, we assumed that the subgroups $\mathcal{T}_i$ satisfy $[\mathcal{K}_{T_i} : \mathcal{T}_i \cap \mathcal{K}_{T_i}] = 1$ and that the conductors $\mathfrak{f}_i \ll 1$ are bounded. This means that

$$\mathrm{vol}([\mathcal{T}_i(g_i, s_i g_i)]) \asymp_F [T_i(\mathbb{A}_F) : \mathcal{T}_i]^{-1}L(1, \chi_{E_i/F})|D_{E_i}|^{1/2}.$$

For homogeneous Hecke sets in the joint setting over F, it is easily verified that Section 7 of [Kha17] generalises directly, so we simply record here the volume result that we need (see Lemma 7.6 of *loc. cit.*). Recall that for $(x, y) \in \mathbb{G}(\mathbb{A})^2, \mathrm{ctr}((x, y)) := x^{-1}y \in \mathbb{G}(\mathbb{A})$.

Definition 4.2.2. *Define the proper continuous function* $\mathfrak{d} : G(\mathbb{A}_{F,f}) \to \mathbb{N}$ *by*

$$\mathfrak{d}_{sf}(h_f) = \prod_{\substack{\nu \nmid \infty \\ \nu \text{ splits } B}} \mathfrak{d}_\nu(h_\nu)$$

Lemma 4.2.3. *Let* $\xi \in (\mathbb{G} \times \mathbb{G})(\mathbb{A}) = (G \times G)(\mathbb{A}_F)$ *with* $\mathrm{ctr}(\xi)_\infty \in K_\infty$*, then*

$$\mathrm{vol}\left(\left[\mathbb{G}^\Delta(\mathbb{A})\xi\right]\right) m_{\mathbb{G}}(\Omega) = \mathfrak{d}_{sf}(\mathrm{ctr}(\xi)_f) \prod_{\substack{\nu : \mathfrak{d}_\nu(\mathrm{ctr}(\xi)_\nu) > 1 \\ \nu \text{ splits } B}} \left(1 + \frac{1}{N\mathfrak{p}_\nu}\right).$$

Chapter 5

The Original Approach to Mixing

In an early version of [EMV10], a proof for a particular case of mixing was proposed for the quotient of the Hamilton quaternion algebra over $\mathbb{Q}$ (i.e. mixing on integer points on spheres). This was removed in the final version - however is recapped by Khayutin in [Kha17, §10.4]. Here, we discuss the result in slightly more detail. However, for arithmetic applications, it is the cases not covered by this approach that have the most significance (i.e. the cases where the class of the twist has no small norm representatives). Therefore we will be relatively brief in our discussion.

We begin by reviewing the set-up of the proof in [EMV10], and give a number of hypotheses in the general case under which some form of mixing can be proven. In the later parts of this section we will then discuss these hypotheses one by one. As set out previously, we consider a sequence of torus orbits $[\mathcal{T}_i(g_i, s_i g_i))] \subset [\mathbb{G}(\mathbb{A})]$. For now we will simply assume that these are all fixed by the right action of $A^+ \subset \mathbb{G}(\mathbb{Q}_S)$, which come from a fixed split maximal torus in $\mathbb{G}(\mathbb{Q}_S)$. As explained previously, it is possible to relax this assumption in general.

5.1 Step-wise Mixing

First, we start with a simple set-theoretic result abstracted from the original version of [EMV10] showing how to prove equidistribution in steps.

Definition 5.1.1. *Given a map $f : A \to B$ of finite sets, define the deviation of an element $b \in B$ to be*

$$\mathrm{dev}_f(b) = \frac{|f^{-1}(b)|\,|B|}{|A|} - 1.$$

Proposition 5.1.2. *For $n \geq 1$, let*

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C$$

be maps of non-empty finite sets with the following properties:

1. *For any $\delta > 0$, as $n \to \infty$,*

$$\frac{1}{|B_n|} \# \{ b \in B_n : |\mathrm{dev}_{f_n}(b)| > \delta \} \to 0.$$

2. *The sequence of maps $\{g_n\}_n$ equidistributes, i.e. for all $c \in C$, as $n \to \infty$,*

$$\frac{|g_n^{-1}(c)|}{|B_n|} - \frac{1}{|C|} \to 0.$$

Then the sequence of maps $\{g_n \circ f_n\}_n$ equidistributes.

Proof. This is proven in Section 3.4 of the earlier version of [EMV10]. For any $1 > \epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall c \in C$,

$$\frac{1}{|B_n|} \# \{ b \in B_n : |\mathrm{dev}_{f_n}(b)| > \epsilon \} < \frac{\epsilon}{2(1-\epsilon)}, \text{ and,}$$
$$\frac{|g_n^{-1}(c)|}{|B_n|} > \frac{1}{|C|} - \frac{\epsilon}{2(1-\epsilon)}.$$

So for each $c \in C$ and $n \geq N$,

$$\begin{aligned}
\frac{\left|(g_n \circ f_n)^{-1}(c)\right|}{|A_n|} &= \sum_{b \in g_n^{-1}(c)} \frac{|f_n^{-1}(b)|}{|A_n|} \\
&\geq \sum_{\substack{b \in g_n^{-1}(c) \\ \left|\mathrm{dev}_{f_n}(b)\right| \leq \epsilon}} \frac{|f_n^{-1}(b)|}{|A_n|} \\
&\geq (1-\epsilon) \sum_{\substack{b \in g_n^{-1}(c) \\ \left|\mathrm{dev}_{f_n}(b)\right| \leq \epsilon}} \frac{1}{|B_n|} \\
&= (1-\epsilon) \left(\frac{|g_n^{-1}(c)|}{|B_n|} - \frac{1}{|B_n|} \# \left\{ b \in g_n^{-1}(c) : |\mathrm{dev}_{f_n}(b)| > \epsilon \right\} \right) \\
&> (1-\epsilon) \frac{1}{|C|} - \epsilon.
\end{aligned}$$

Therefore for each $c \in C$,

$$\lim_{n \to \infty} \frac{\left|(g_n \circ f_n)^{-1}(c)\right|}{|B_n|} \geq \frac{1}{|C|}.$$

This being true for every element of C implies the opposite inequality by examining the complement $C \setminus \{c\}$, and so we have equality. ∎

This is applied in the method of Ellenberg-Michel-Venkatesh by taking the quotient of $\mathbb{G}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{A}_f)$ by some open compact subgroup $U < \mathbb{G}(\mathbb{A}_f)$ and therefore moving to a finite situation. Then, the sets A_n above are the images of the toral sets, and the sets B_n are the images of intermediate homogeneous Hecke sets inside $\mathbb{G} \times \mathbb{G}$. The above proposition then reduces the question of equidistribution of the torus sets (in this finite quotient) to two questions - firstly the equidistribution of the Hecke sets, and secondly the study of deviations of the torus sets in the Hecke sets.

In the scenario of [EMV10], the only possible Hecke orbits are those from the diagonal subgroup. However, the approach may apply in some limited cases to prove single equidistribution. For this, we note that there is an obvious generalisation of the Proposition above.

Proposition 5.1.3. *Fix $k \geq 1$. For $n \geq 1$, let*

$$A_n^{(0)} \xrightarrow{f_n^1} A_n^{(1)} \xrightarrow{f_n^2} \dots \xrightarrow{f_n^k} A_n^{(k)} \xrightarrow{g_n} C$$

be maps of non-empty finite set with the following properties:

1. *For each $1 \leq r \leq k$, and any $\delta > 0$, as $n \to \infty$,*

$$\frac{1}{|A_n^{(r)}|} \# \left\{ x \in A_n^{(r)} : \left| \mathrm{dev}_{f_n^r}(x) \right| > \delta \right\} \to 0.$$

2. *The sequence of maps $\{g_n\}_n$ equidistributes.*

Then the sequence of maps $\left\{ g_n \circ f_n^k \circ \dots \circ f_n^1 \right\}_n$ equidistributes.

Proof. Induction on k. □

This could possibly be used in cases where there is a filtration of intermediate homogeneous sets between the torus sets and the whole space, and each time the deviations could be studied between the sets at each layer. This is not something we pursue here however.

5.2 Associated Buildings

The original proof of mixing uses Bruhat-Tits buildings coming from the Hecke correspondences containing these toral sets. Consider a single toral set, which we will label as $\left[\mathbb{T}(\mathbb{A})^{+,\Delta} (g, sg) \right]$ for $g \in \mathbb{G}(\mathbb{A}), s \in \mathbb{T}(\mathbb{A})$, where $\mathbb{T}$ is a maximal rank anisotropic torus $\mathbb{T} \leq \mathbb{G}$. This is contained in the homogeneous Hecke set $\left[\mathbb{G}(\mathbb{A})^{+,\Delta} (g, sg) \right]$.

To use the step-wise mixing approach from Proposition 5.1.2, we need to reduce to a case of finite sets. To do this, we pick an open compact subset $U = \prod_p U_p \subset \mathbb{G}(\mathbb{A}_f)$, and we will reduce the inclusions

$$\left[\mathbb{T}\left(\mathbb{A}\right)^{+,\Delta}(g, sg)\right] \to \left[\mathbb{G}(\mathbb{A})^{+,\Delta}(g, sg)\right] \to \left[\left(\mathbb{G} \times \mathbb{G}\right)(\mathbb{A}_f)\right]$$

modulo $U \times U$. We can also pull back to the simply connected cover, in which case the middle set becomes

$$\mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/\pi^{-1}\left(gUg^{-1} \cap sgUg^{-1}s^{-1}\right),$$

where $\pi : \mathbb{G}^{sc} \to \mathbb{G}$. The equidistribution of the Hecke subvarieties (which provides the equidistribution of the functions g_n in this set up for Proposition 5.1.2) is a well-known result ([COU01]).

5.3 The Spectral Gap

In [COU01], a general construction for the spectral gap for Hecke operators is given for connected almost-simple simply-connected linear algebraic groups. As mentioned in the Remarks following Theorem 1.7 of that paper, while they assume $G(\mathbb{R})$ is non-compact, there is no need for this assumption, and the results hold also at a finite non-compact place. Since we are assuming that the group $\mathbb{G}$ splits completely at the places $p \in S$ for which we are using the torus action $\mathbb{T}(\mathbb{Q}_S)^+$, their results simplify somewhat. In particular, it is only the case of $\mathrm{SL}_1(D_F)$ for a quaternion algebra D_F over F for which Clozel-Oh-Ullmo require bounds towards the Ramanujan Conjecture. Otherwise, the bounds come from previous results of Oh ([Oh02]).

Here we combine a number of observations in [COU01] with the simplification of splitting completely at the places $p \in S$ to get a strong statement of the norm gap:

Theorem 5.3.1 (from [COU01]). *Let G be an absolutely almost-simple simply-connected algebraic group over a totally real field F such that $G(F_\infty)$ is non-compact. Suppose that S is a set of finite places of $\mathbb{Q}$ such that F splits completely over each $p \in S$ and furthermore G splits over each place of F lying above $p \in S$. Fix a maximal rank split torus $A = \prod_{\mathfrak{p}|p\in S} A_\mathfrak{p} \leq G(F_S)$, and an open compact subgroup $U = \prod_{\mathfrak{p}\in\Sigma_F^\infty} U_\mathfrak{p} \leq G\left(\widehat{F}\right)$ contained in a maximal open compact subgroup $K = \prod_{\mathfrak{p}\in\Sigma_F^\infty} K_\mathfrak{p}$. For any regular $a \in A$, consider the Hecke operator*

$$T_a^0 : L_0^2(G(F) \setminus G(\mathbb{A}))^U \to L_0^2(G(F) \setminus G(\mathbb{A}))^U$$

acting on the orthogonal complement to the constant functions.

Choose a maximal strongly orthogonal system[1] $\mathcal{S}$ *for the split form of* G*, and consider this as a maximal strongly orthogonal system at each place* $\mathfrak{p}|p \in S$*. For each* $\mathfrak{p}|p \in S$*, define*

$$n_{\mathcal{S},\mathfrak{p}}(a) = -\frac{1}{2}\sum_{\alpha\in\mathcal{S}} \log_p |\alpha(a_{\mathfrak{p}})|_{\mathfrak{p}} > 0.$$

Then, for all $\epsilon > 0$*,*

$$||T_a^0|| \leq C(\epsilon)\,[K_S : U_S] \prod_{\mathfrak{p}|p\in S} p^{-n_{\mathcal{S},\mathfrak{p}}(a)(1-\epsilon-\theta)}$$

where the constant $C(\epsilon)$ *does not depend on* U *or* a*, and the constant* θ *is given by*

$$\theta = \begin{cases} 0, & \textit{if } \mathrm{rnk} G_{\overline{F}} \geq 2 \\ \textit{Best bound towards GRC for } \mathrm{SL}_{2,\mathbb{Q}_p} \forall p \in S, & \textit{if } \mathrm{rnk} G_{\overline{F}} = 1. \end{cases}$$

Proof. This is all contained in [COU01]. To apply this to the Hecke operator attached to a we are using the translation of the Hecke operators to the adelic setting as in Section 2. The S-adic nature over the number field F is explained in the remarks after Theorem 1.7. The exact input of the Ramanujan is Section 3.2 in terms of decay of matrix coefficients. We have also used the slightly stronger version of the bound from Section 4.2 noting that for us the elements a have bounded support in the primes over S and so

$$\prod_{\mathfrak{p}\in R_\Gamma(a)} \max_{d\in\Omega_{\mathfrak{p}}} \left[K_{\mathfrak{p}} : dK_{\mathfrak{p}}d^{-1}\right]$$

is uniformly bounded as $R_{\Gamma,a}$ is contained in the set of primes dividing S. □

In many cases, as explained in [COU01], we may relax the condition that $G(F_\infty)$ is non-compact by a transfer from G to an inner form G' with $G'(F_\infty)$ non-compact. In particular, the same results apply to $\mathrm{SL}_1(D_F)$ and the special unitary groups $SU_{n,F}$. Also, note that by Theorem 9.3 of [GJ78], we can assume $\theta \leq 1/2$.

[1] A strongly orthogonal system is a subset $\mathcal{S} \subset \Phi^+$ of positive roots such that for any two distinct $\alpha, \beta \in \mathcal{S}$, $\alpha \pm \beta \notin \Phi$. Such a system is called maximal if the coefficient of each simple root in the sum $\sum_{\alpha\in\mathcal{S}} \alpha$ is not less that the one in $\sum_{\alpha\in\mathcal{S}'} \alpha$ for any other strongly orthogonal system $\mathcal{S}'$ of Φ. See [Oh98] for a construction for each root system.

5.4 Effective Equidistribution in the Homogeneous Hecke Set

We now prove effective equidistribution of the torus orbits within the homogeneous Hecke sets. This follows the method of [EMV10] with the alternative method of [Kha15] for converting a norm gap and Bowen ball estimates to effective equidistribution.

The essential diagram is as follows: suppose that we lift the torus $\mathbb{T} < \mathbb{G}$ to a torus $\widetilde{\mathbb{T}} < \mathbb{G}^{sc}$, then we wish to analyse

$$\begin{array}{ccc} \widetilde{\mathbb{T}}(\mathbb{Q}) \setminus \widetilde{\mathbb{T}}(\mathbb{A}_f)/\widetilde{\mathbb{T}}(\mathbb{A}_f) \cap U & \longrightarrow & \mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U^{(0,l)} \\ & \searrow & \downarrow \\ & & \mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U \end{array}$$

Recall that $U^{(0,l)} := \bigcap_{i=0,\ldots,l-1} a^i U a^{-i}$. The result we wish to prove is a bound on the number of points in $\mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U$ with large deviations. The sketch proof of this result is as follows: if there is a set of points with large preimages, we would expect many walks on the building to pass through this set many times - and this contradicts the 'spreading out' property of the Hecke action, which averages over the points in a walk. The space in the top right of the diagram represents the space of walks of length l on the building in the bottom right. First, let us discuss the spreading out Hecke action and the implication of this for walks on the buildings.

For walks to have the properties that we would like, we require the quotient by U_S to have property (M), a property introduced in [Kha15]. In particular, we have

Proposition 5.4.1. *Suppose that for each $p \in S$, the element a_p^{-1} acts on the apartment associated to A_p in the affine Bruhat-Tits building Δ_p associated to $\mathbb{G}(\mathbb{Q}_p)$ sending a fixed special vertex $v_{0,p}$ to a different special vertex $v_{1,p}$ which does not have a shared wall with $v_{0,p}$ (this assumption is fulfilled when a is $\mathbb{Q}_p$-regular and does not belong to a compact subgroup). Let $U_p < \mathbb{G}(\mathbb{Q}_p)$ be the arrow subgroup defined in [Kha15, §4.3] corresponding to a_p^{-1} and $v_{0,p}$. Then*

$$\mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U^S \to \mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U$$

has property (M) with respect to the right action by $a \in \mathbb{G}(\mathbb{Q}_S)$.

Proof. It is clear that [Kha15, Lemma 4.2] holds in exactly the same way for $K < \mathbb{G}(\mathbb{Q}_S)$ where S is allowed to have more than one element. Now, Proposition 4.7 of

[Kha15] tells us that there exist $\omega_1^p, ..., \omega_{k_p}^p \in U_p^{(-\infty,0)}$ such that

$$U_p = \bigsqcup_{j=1}^{k_p} \omega_j^p U_p^{(0,1)}.$$

Therefore, if we set $\omega_{\underline{i}}^S = \left(\omega_{i_{p_1}}^{p_1}, ..., \omega_{i_{p_{|S|}}}^{p_{|S|}}\right)$ for $\underline{i} \in \{1, ..., k_{p_1}\} \times ... \times \left\{1, ..., k_{p_{|S|}}\right\}$, then

$$U_S = \bigsqcup_{\underline{i}} \omega_{\underline{i}}^S U_S^{(0,1)}.$$

Now the same proof as Corollary 4.3 in [Kha15] proves that the assumptions of the S-adic version of Lemma 4.2 are satisfied, and so the quotient has property (M). □

Define $X^l := \mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U^{(0,l)}$. We claim that an element of X^l is precisely a length l walk on $X = \mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U$ with respect to the action from a^{-1}. The map is given by

$$x \mapsto \left(x, xa^{-1}, xa^{-2}, ..., xa^{-l}\right).$$

This is

1. Well-defined: since for any $u \in U^{(0,l)} = \bigcap_{i=0,...,l} a^{-i}Ua^i$ and any $i = 0, ..., l$, we have

$$\mathbb{G}^{sc}(\mathbb{Q})xua^{-i}U = \mathbb{G}^{sc}(\mathbb{Q})xa^{-i}U.$$

2. Bijective: By induction. Suppose the claim is true for paths of length $l \geq 0$. Then, the paths of length $l+1$ map to the paths of length l with degree $\left[U : U^{(0,1)}\right]$ by definition of the paths. So, it remains to show that given a fixed $\mathbb{G}^{sc}(\mathbb{Q})xU^{(0,l)}$, all of the paths of length $l+1$ extending this are constructed from some $\mathbb{G}^{sc}(\mathbb{Q})xuU^{(0,l+1)}$ for some $u \in U^{(0,l)}$. Corollary 4.3 of [Kha15] gives us that

$$U^{(0,l)} = \bigsqcup_{j=1}^{k} a^{-l}\omega_j a^l U^{(0,l+1)}.$$

Therefore, we can consider the path attached to $\mathbb{G}^{sc}(\mathbb{Q})xa^{-l}\omega_j a^l U^{(0,l+1)}$. Using the fact that $a^{-i}\omega_j a^i \in U$ for all $i \geq 0$, we see that the corresponding path is

$$(x, xa^{-1}, ..., xa^{-l}, xa^{-l}\omega_j a^{-1})$$

Thus every neighbour of xa^{-l} is accounted for, and we see that every path of length $l+1$ is achieved. Since the degree of both maps are $\left[U : U^{(0,1)}\right] = \left[U^{(0,l)} : U^{(0,l+1)}\right] = k := \prod_{p \in S} k_p$, we are done.

Now take a subset $\mathcal{B} \subset X := X^0$. We wish to get an exponential decay bound on the number of paths that spend a disproportionately large amount of time in the set $\mathcal{B}$, and compare this with the lower bounds of [EMV10]. For this we use the large deviation estimates from Theorem 3.2 of [Kha15]. Combining the exponential decay with the lower bound gives a contradiction to the assumption that the deviations do not go to zero. The following is an informal result:

Proposition 5.4.2. *The mixing conjecture holds for quaternion algebras over a totally real field F provided that for $\mathfrak{N}_i := \min_{\substack{\mathfrak{a} \subset \Lambda \\ [\mathfrak{a}] = [\mathfrak{s}_i]}} N_{K/F}\mathfrak{a}$,*

$$N_{F/\mathbb{Q}}\mathfrak{N}_i \ll |D_i|^{\frac{[F:\mathbb{Q}]}{2([F:\mathbb{Q}]+2)}-\epsilon}$$

for any $\epsilon > 0$.

Sketch Proof. Firstly, we are proving this equidistribution via Proposition 5.1.2, and, as already noted, the equidistribution for the functions g_n is Hecke equidistribution (known from [COU01] for example). Thus we just need to bound the points with high deviations.

Let $Y = \mathbb{G}^{sc}(\mathbb{Q}) \setminus \mathbb{G}^{sc}(\mathbb{A}_f)/U^S$, and $\varphi = 1_{\mathcal{B}} \circ \pi_X : Y \to \{0,1\}$ indicate the paths starting in $\mathcal{B}$. The action of a on Y gives the projection $Y \to X$ property (M) as explained above. The corresponding Hecke operator T_a^0 is precisely the one in Theorem 5.3.1, and therefore we let $\lambda = ||T_a^0||$ as in that Theorem.

Let $\mathcal{B} = \{x \in X : \mathrm{dev}(x) > \delta\}$. Let $\mu, \eta > 0$. The proportion of paths of length l spending $\geq m(\mathcal{B}) + \mu$ proportion of time in $\mathcal{B}$ is precisely given by

$$m\left(y \,\middle|\, \frac{1}{l+1}\sum_{i=0}^{l} \varphi(a^n \cdot y) \geq m(\mathcal{B}) + \mu\right).$$

By Theorem 3.2 of [Kha15], this quantity can be bounded above by

$$\frac{\log \rho}{\log \lambda} e^{D(m(\mathcal{B})+\mu||m(\mathcal{B}))} \exp\left[-(l+1)(-\log\lambda)\frac{D(m(\mathcal{B})+\mu||\rho+(1-\rho)m(\mathcal{B})}{-\log\rho}\right], \tag{5.1}$$

(here $D(p||q) = -p\log(q/p) - (1-p)\log((1-q)/(1-p))$ is the Kullback-Leibler divergence, a measure of the difference between p and q), if

$$\rho = \theta^k \leq \frac{\eta}{1-m(\mathcal{B})}.$$

On the other hand, in [EMV10], it is proven that for any $\mathfrak{p}|p \in S$, if $m(B_\delta) \geq \eta$, $\mu = \delta\eta/2$ and

$$N\mathfrak{N}_i N\mathfrak{p}^{n_{S,\mathfrak{p}}(a)} \leq |D_i|^{1/2+o(1)}, \tag{5.2}$$

then we can choose $l \asymp \log |D|$ such that the proportion of paths of length l which spend $\geq m(\mathcal{B}) + \mu$ proportion of time in $\mathcal{B}$ *and* lie in the image of $\mathbb{T}_i$ is $\gg_{\epsilon,\delta,\eta} |D|^{-\epsilon}$ for any $\epsilon > 0$.

Under these assumptions, and the additional assumption that $\lambda_i \to 0$, we see that we can eventually set $\rho = \lambda$, and the quantity of equation (5.1) is

$$\ll \exp\left[-A(l+1)\right]$$

for some positive constant $A > 0$ independent of i. Therefore to get a contradiction (to $m(B_\delta) \geq \eta$), we must arrange that simultaneously $\lambda_i \to 0$ and equation (5.2) holds.

For $\mathfrak{p}|\mathfrak{R}_i$, we get

$$[K_\mathfrak{p} : U_\mathfrak{p}] = N\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}\mathfrak{R}_i}\left(1 + \frac{1}{N\mathfrak{p}}\right)$$

by the volume computation of the homogeneous Hecke sets. Therefore, as we have multiple completely split primes, we can assume $N\mathfrak{p}^{[F:\mathbb{Q}]\mathrm{ord}_\mathfrak{p}\mathfrak{R}_i} \leq N\mathfrak{R}_i^{1/2}$, and under the assumption that

$$N\mathfrak{p}^{n_{S,\mathfrak{p}}(a)} \gg N\mathfrak{R}_i^{1/[F:\mathbb{Q}](1-\theta-\epsilon)}$$

we see that $\lambda \to 0$. Using the bound $\theta = 1/2$ for the Ramanujan Conjecture (noting that we required Jacquet-Langlands to move to the quaternion setting), this is compatible with equation (5.2) if

$$\mathfrak{R}_i \ll |D_i|^{\frac{[F:\mathbb{Q}]}{4+2[F:\mathbb{Q}]}-\epsilon}.$$

□

Chapter 6

Expansions of the Correlations

In this section, we expand the correlation and relate it to analytically tractable sums.

6.1 The Geometric Expansion

The aim of this section is to perform the general geometric expansion of an algebraic torus measure against an arbitrary homogeneous measure. Since we are working with potentially smaller torus orbits, we must compare $[\mathcal{T}g]$ with $[\mathbb{L}(\mathbb{A})^+h]$ where $\mathcal{T} > \mathbb{T}(\mathbb{A})^+ := \mathbb{G}(\mathbb{A})^+ \cap \mathbb{T}(\mathbb{A})$ and $\mathbb{L}(\mathbb{A})^+ := \mathbb{L}(\mathbb{A}) \cap \mathbb{G}(\mathbb{A})^+$. We also assume that $\mathbb{T}(\mathbb{Q}) < \mathcal{T}$. For now, we are just considering a comparison of $\mathbb{T}$ and $\mathbb{L}$ inside a group $\mathbb{G}$, which we will then apply separately to the joint and Kuga-Sato specific cases.

6.1.1 The General Expansion

Let μ be the algebraic (probability) measure associated to $[\mathcal{T}g]$ and ν to $[\mathbb{L}(\mathbb{A})^+h]$. The quantity we wish to analyse is

$$\mathrm{Corr}(\mu,\nu)\,[f] = \int_{[\mathbb{G}(\mathbb{A})]}\int_{[\mathbb{G}(\mathbb{A})]}\sum_{\gamma\in\mathbb{G}(\mathbb{Q})} f(y^{-1}\gamma x)d\mu(x)d\nu(y).$$

Setting $f_0(z) = f(h^{-1}zg)$, and $W_{\mathbb{Q}}^+ = \mathbb{L}(\mathbb{Q})^+ \setminus \mathbb{G}(\mathbb{Q})/\mathbb{T}(\mathbb{Q})$, we get

$$\begin{aligned}\mathrm{Corr}(\mu,\nu)[f] &= \int_{[\mathcal{T}]}\int_{[\mathbb{L}(\mathbb{A})^+]}\sum_{\gamma\in\mathbb{G}(\mathbb{Q})} f_0(y^{-1}\gamma x)dydx \\ &= \sum_{[\gamma]\in W_{\mathbb{Q}}^+}\int_{[\mathcal{T}]}\int_{[\mathbb{L}(\mathbb{A})^+]}\sum_{\delta\in\mathbb{T}(\mathbb{Q})\gamma\mathbb{L}(\mathbb{Q})^+} f_0(y^{-1}\delta x)dydx\end{aligned}$$

Here, dx and dy refer to the normalised Haar measures on $[\mathcal{T}]$ and $[\mathbb{L}(\mathbb{A})^+]$ respectively. Let $\mathbb{M} := \mathbb{T}\times\mathbb{L}$ act on $\mathbb{G}$ via left multiplication for $\mathbb{L}$ and right multiplication by the inverse for $\mathbb{T}$, and for any closed subgroup let $N < \mathbb{M}(\mathbb{A})$, $N^\dagger := N \cap (\mathcal{T} \times \mathbb{L}(\mathbb{A})^+)$. Clearly these are motivated by [Kha17], see Definition 8.10, except that we require additional flexibility in our finite index subgroup of the torus. Also, let $\mathbb{M}_\gamma$ for $\gamma \in \mathbb{G}(\mathbb{Q})$ denote the stabiliser of γ in $\mathbb{M}$. Using these, we get

$$\begin{aligned}
\mathrm{Corr}(\mu,\nu)[f] &= \sum_{[\gamma]\in W_\mathbb{Q}^+} \int_{[\mathbb{M}(\mathbb{A})]^\dagger} \sum_{\delta\in\mathbb{M}(\mathbb{Q})^\dagger\cdot\gamma} f_0(m^{-1}\cdot\delta)dm \\
&= \sum_{[\gamma]\in W_\mathbb{Q}^+} \int_{\mathbb{M}_\gamma(\mathbb{Q})^\dagger\backslash\mathbb{M}(\mathbb{A})^\dagger} f_0(m^{-1}\cdot\gamma)dm \\
&= \sum_{[\gamma]\in W_\mathbb{Q}^+} \mathrm{vol}\left(\mathbb{M}_\gamma(\mathbb{Q})^+ \setminus \mathbb{M}_\gamma(\mathbb{A})^\dagger\right) \int_{\mathbb{M}_\gamma(\mathbb{A})^\dagger\backslash\mathbb{M}(\mathbb{A})^\dagger} f_0(m^{-1}\cdot\gamma)dm.
\end{aligned}$$

In fact, this can be simplified slightly since many of the subgroups $\mathbb{M}_\gamma$ are conjugate inside $\mathbb{M}$ for different $\gamma \in W_\mathbb{Q}^+$. In fact, $\mathbb{M}(\mathbb{Q})$ acts on $W_\mathbb{Q}^+$ with orbits given by

$$W_\mathbb{Q} = \mathbb{L}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{Q})/\mathbb{T}(\mathbb{Q}),$$

and if two elements of $W_\mathbb{Q}^+$ lie in the same orbit, their stabiliser groups $\mathbb{M}_\gamma$ are conjugate, and so have the same volume. Thus, we can separate the formula into two summations as follows:

$$\mathrm{Corr}(\mu,\nu)[f] = \sum_{[\gamma]\in W_\mathbb{Q}} \mathrm{vol}(\mathbb{M}_\gamma^\dagger) \sum_{\delta\in\mathbb{M}(\mathbb{Q})\gamma\subset W_\mathbb{Q}^+} \int_{\mathbb{M}_\delta(\mathbb{A})^\dagger\backslash\mathbb{M}(\mathbb{A})^\dagger} f_0(m^{-1}\cdot\delta)dm$$

Finally, since $\delta = \gamma'\gamma$ for some $\gamma' \in \mathbb{M}(\mathbb{Q})$, we can write $m^{-1}\cdot\delta = \gamma'\left(\mathrm{Ad}_{(\gamma')^{-1}}m\right)^{-1}\cdot\gamma$. Using the fact that Haar measure on $\mathbb{M}(\mathbb{A})^\dagger$ is invariant under conjugation by $\mathbb{M}(\mathbb{A})$, we get the final form of the geometric expansion:

Proposition 6.1.1. *The correlation has the form*

$$\mathrm{Corr}(\mu,\nu)[B] = \sum_{[\gamma]\in W_\mathbb{Q}} \mathrm{vol}\left(\mathbb{M}_\gamma^\dagger\right) \sum_{\delta\in\mathbb{M}(\mathbb{Q})\gamma\subset W_\mathbb{Q}^+} \mathrm{RO}_\delta(B)$$

where for $\delta \in W_\mathbb{Q}^+$,

$$\mathrm{RO}_\delta(B) = \int_{(\mathbb{M}_\delta\backslash\mathbb{M})(\mathbb{A})^\dagger} 1_{hBg^{-1}}\left(m^{-1}\cdot\delta\right)dm$$

Notice that the presence of the inner sum here is simply due to the extra flexibility of smaller subgroups. If we had decided to only consider the full homogeneous sets, then there would only be the summation over $W_\mathbb{Q}$.

In general, there appears to be the following expectation:

Assumption 6.1.2. *For any $[\gamma] \in W_{\mathbb{Q}}$ such that $\mathbb{M}_{\gamma}^{\dagger}(\mathbb{A})$ is non-compact, the contribution of the corresponding terms to $\mathrm{Corr}(\mu,\nu)[B]$ should be negligible.*

This assumption appears to be compatible with GIT in the sense that such $[\gamma]$ correspond to non-stable points of the GIT quotient, which are somehow singular. The non-compactness of the stabilisers of these points should give a lower dimension for the orbit, and therefore the assumption appears to be sensible. Indeed, it follows in both the joint CM case (see Lemma 8.14 of [Kha17]) and the Kuga-Sato case (see Proposition 4.6 of [Kha19b]), both of which immediately apply in the totally real case also. The joint CM case relies on the GIT analysis of Section 6 of [Kha17], which is true verbatim with $\mathbb{Q}$ replaced by F, and so we will say nothing more about this section.

6.1.2 Kuga-Sato Correlations

In the Kuga-Sato case, we set $G = \mathrm{SL}_{2,F} \ltimes \mathbb{G}_{a,F}^2$ and $L = \mathrm{SL}_{2,F}$, the correlation expansion of Proposition 6.1.1 from the previous section is particularly clean. We have a toral measure μ attached to $[\mathcal{T}(l,x)]$ and a Hecke measure ν attached to $[\mathbb{G}(\mathbb{A})^+(h,y)]$.

Proposition 6.1.3. *As $i \to \infty$, the definition of strictness of toral sets in the Kuga-Sato setting implies Assumption 6.1.2 holds, so we can assume the relative orbital integral attached to the identity $0 \in V$ is zero. Then*

$$\mathrm{Corr}(\mu,\nu)[B] = \sum_{\substack{0 \neq v \in T(F)\backslash V(F) \\ \varkappa \in \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+}} \mathrm{RO}_{v,\varkappa}(B)$$

where

$$\mathrm{RO}_{\varkappa,v}(B) = \int_{\mathbb{G}(\mathbb{A})^+ \times \mathcal{T}} 1_{(h,y)^{-1}B(l,x)}(g^{-1}\varkappa t, g^{-1}v)\, d(g,t).$$

6.1.3 Joint CM Correlations

We also get a result in the joint CM case.

Proposition 6.1.4. *Let μ be the periodic measure on the joint homogeneous toral set $[\mathcal{T}(g,sg)]$ and ν the period measure on $\left[\mathbb{G}^{\Delta}(\mathbb{A})^+(\xi_1,\xi_2)\right]$. Set $B' = \xi_1 B g^{-1} \times$*

$\xi_2 B g^{-1} s^{-1}$, *then*

$$\begin{aligned} \mathrm{Corr}(\mu,\nu)[B] &= \int_{[\mathbb{G}(\mathbb{A})^+]} \int_{[\mathcal{T}]} K_{B'}(l,t)dldt \\ &= \sum_{[\gamma]\in W_{\mathbb{Q}}} \sum_{\varkappa\in\pi_{\mathbb{G}}(\mathbb{M}_\gamma(\mathbb{Q}))\backslash\mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+} \mathrm{vol}(\mathbb{M}_\gamma)\mathrm{RO}_{\gamma,\varkappa}(B) \\ \mathrm{RO}_{\gamma,\varkappa}(B) &:= \int_{\mathbb{M}_\gamma(\mathbb{A})^\dagger\backslash\mathbb{M}(\mathbb{A})^\dagger} 1_{B'}\left((\varkappa l)^{-1}\gamma t\right) d(l,t) \\ \mathrm{vol}(\mathbb{M}_\gamma) &:= m_{\mathbb{M}_\gamma(\mathbb{A})^\dagger}\left(\mathbb{M}_\gamma(\mathbb{Q})^\dagger \setminus \mathbb{M}_\gamma(\mathbb{A})^\dagger\right) \end{aligned}$$

where the Haar measures on $\mathbb{M}_\gamma(\mathbb{A})^\dagger \setminus \mathbb{M}(\mathbb{A})^\dagger$ and $\mathbb{M}_\gamma(\mathbb{A})^\dagger$ are mutually normalised. Recall that for a subgroup $N \leq \mathbb{M}(\mathbb{A}) := \mathbb{G}(\mathbb{A}) \times \mathbb{T}(\mathbb{A})$, we define $N^\dagger = N \cap (\mathbb{G}(\mathbb{A})^+ \times \mathcal{T})$.

Proof. The assumption that $\mathbb{T}(\mathbb{Q}) \leq \mathcal{T}$ implies that

$$\mathbb{M}_\gamma(\mathbb{Q}) \setminus \mathbb{M}(\mathbb{Q})/\mathbb{M}(\mathbb{Q})^\dagger \cong \pi_{\mathbb{G}}(\mathbb{M}_\gamma(\mathbb{Q})) \setminus \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+.$$

□

As mentioned previously, in this geometric expansion we may remove the points with non-compact stabiliser, via the Lemma below, which follows exactly as Lemma 8.14 of [Kha17]. Recall that for $(x,y) \in \mathbb{G}(\mathbb{A})^2, \mathrm{ctr}((x,y)) := x^{-1}y \in \mathbb{G}(\mathbb{A})$.

Assumption 6.1.5.

$$g^{-1}\mathbb{T}(\mathbb{Q})sg \cap B^{-1}\mathrm{ctr}(\xi)B = \emptyset$$

Lemma 6.1.6. *Suppose that Assumption 6.1.5 holds. Then for all $\gamma \in (\mathbb{G}\times\mathbb{G})(\mathbb{Q})$ such that $\mathbb{M}_\gamma(\mathbb{A})^\dagger$ is not compact,*

$$\mathrm{RO}_{\gamma,\varkappa}(B) = 0, \forall \varkappa \in \pi_{\mathbb{G}}(\mathbb{M}_\gamma(\mathbb{Q})) \setminus \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+.$$

Furthermore, this occurs precisely when $\mathrm{ctr}(\gamma) \in \mathbb{T}(\mathbb{Q})$.

In Proposition 7.8 of [Kha19b], it is shown that Assumption 6.1.5 holds for all $(\mathbb{T}_i, s_i, g_i)$ with $i \gg 1$ in a sequence of tori on $PB^\times$ such as those considered in our main Theorem (Theorem 2.6.5), since in the conditions of that Theorem we assumed in particular that

$$\mathfrak{R}_i \to \infty \text{ as } i \to \infty.$$

Corollary 6.1.7. *If Assumption 6.1.5 holds, then*

$$\mathrm{Cor}[\mu,\nu](B) = \sum_{\substack{[\gamma]\in W_{\mathbb{Q}} \\ \mathrm{ctr}(\gamma)\notin\mathbb{T}(\mathbb{Q})}} \frac{1}{\#\mathbb{M}_\gamma(\mathbb{Q})} \sum_{\varkappa\in\mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+} \mathrm{RO}_{\gamma,\varkappa}(B).$$

In this sum,

$$\mathrm{RO}_{\gamma,\varkappa}(B) = \int_{\mathbb{M}(\mathbb{A})^\dagger} 1_{B'}\left((\varkappa l)^{-1}\gamma t\right) d(l,t).$$

Recall that $B' := \xi_1 B g^{-1} \times \xi_2 B g^{-1} s^{-1}$*. Furthermore,*

$$\#\mathbb{M}_\gamma(\mathbb{Q}) = \begin{cases} 1, & \textit{if } \mathrm{ctr}(\gamma) \notin N_{\mathbb{G}}\mathbb{T}(\mathbb{Q}) = N_G T(F) \\ 2, & \textit{if } \mathrm{ctr}(\gamma) \in w_{\mathbb{T}}\mathbb{T}(\mathbb{Q}). \end{cases}$$

Proof. This follows in precisely the same manner, since we can choose the volume 1 Haar measure on the stabiliser in the case that it is compact. □

As in all cases, we can split the relative orbital integrals into archimedean and non-archimedean parts in the obvious way,

$$\mathrm{RO}_{\gamma,\varkappa}(B) = \mathrm{RO}^f_{\gamma,\varkappa}(B)\mathrm{RO}^\infty_{\gamma,\varkappa}(B),$$

and the following bound on the archimedean part is immediately generalised from the rational case.

Lemma 6.1.8. *Let* $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{G}\times\mathbb{G})(\mathbb{Q})$ *and* $\varkappa \in \mathbb{G}(\mathbb{Q})$*. Assume that* $B_\infty = \Omega_\infty$ *is a connected, compact, symmetric and* $\mathrm{Ad}K_\infty$*-invariant identity neighbourhood in* $\mathbb{G}(\mathbb{R})$*. Then* $RO^\infty_{\gamma,\varkappa}(B) = 0$ *if* $\mathrm{ctr}(\gamma) \notin g_\infty\Omega_\infty\mathrm{ctr}(\xi)_\infty\Omega_\infty g_\infty^{-1}$*, and*

$$\mathrm{RO}^\infty_{\gamma,\varkappa}(B) \le m_{\mathbb{T}(\mathbb{R})}(\mathbb{T}(\mathbb{R}))m_{\mathbb{G}(\mathbb{R})^+}\left(\xi_{1,\infty}\Omega_\infty^2\xi_{1,\infty}^{-1} \cap \xi_{2,\infty}\Omega_\infty^2\xi_{2,\infty}^{-1}\right)$$

otherwise.

Proof. This is identical to Lemma 8.18 of [Kha19a] since we have made the assumption that $\mathcal{T}_\infty = \mathbb{T}(\mathbb{R})$. □

The non-archimedean relative orbital integrals can be bounded by certain 'intersection numbers'. For convenience, we repeat that for any subset $N \subset \mathbb{G}(\mathbb{A}) \times \mathbb{T}(\mathbb{A})$, we write $N^\dagger := N \cap (\mathbb{G}(\mathbb{A})^+ \times \mathcal{T})$. We also define $N^{\mathcal{T}} = N \cap (\mathbb{G}(\mathbb{A}) \times \mathcal{T})$.

Proposition 6.1.9. *For* $\gamma \in (\mathbb{G}\times\mathbb{G})(\mathbb{Q})$*, define*

$$\mathcal{S}_\gamma := \left\{(l,t) \in \mathbb{M}(\mathbb{A}_f) : l^{-1}\gamma t \in B'_f\right\} \subset \mathbb{M}(\mathbb{A}_f).$$

This set is right-invariant under $B_{\mathbb{M},f}$, defined by

$$\begin{aligned} B_{\mathbb{G},f} &:= \xi_{1,f} B_f \xi_{1,f}^{-1} \cap \xi_{2,f} B_f \xi_{2,f}^{-1}, \\ B_{\mathbb{T},f} &:= g_f B_f g_f^{-1} \cap \mathbb{T}(\mathbb{A}_f), \\ B_{\mathbb{M},f} &:= B_{\mathbb{G},f} \times B_{\mathbb{T},f}. \end{aligned}$$

Let $N_{[\gamma]}$ be the number of orbits of $B_{\mathbb{M},f}^{\mathcal{T}}$ on $\mathcal{S}_\gamma^{\mathcal{T}}$. Then

$$\sum_{\varkappa \in \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+} \mathrm{RO}_{\gamma,\varkappa}^f(B) \leq 2^{[F:\mathbb{Q}]} \left|\mathrm{Cl}_F[2]\right| m_{\mathbb{G}(\mathbb{A}_f)^+}(B_{\mathbb{G},f}^\dagger) m_{\mathcal{T}}(B_{\mathbb{T},f}^\dagger) N_{[\gamma]}.$$

Proof. By definition,

$$\mathrm{RO}_{\gamma,\varkappa}^f(B) := m_{\mathbb{M}(\mathbb{A})^\dagger}\left(\mathcal{S}_{\varkappa^{-1}\gamma}^\dagger\right).$$

The set $\mathcal{S}_{\varkappa^{-1}\gamma}^\dagger$ is right-invariant under $B_{\mathbb{M},f}^\dagger$ with finitely many orbits, and so

$$\mathrm{RO}_{\gamma,\varkappa}^f(B) = m_{\mathbb{G}(\mathbb{A}_f)^+}(B_{\mathbb{G},f}^\dagger) m_{\mathcal{T}}(B_{\mathbb{T},f}^\dagger) N_{\gamma,\varkappa},$$

where $N_{\gamma,\varkappa}$ is the number of $B_{\mathbb{M},f}^\dagger$-orbits in $\mathcal{S}_{\varkappa^{-1}\gamma}^\dagger$. Now,

$$\varkappa \mathcal{S}_{\varkappa^{-1}\gamma}^\dagger \subset \varkappa \mathcal{S}_{\varkappa^{-1}\gamma} = \mathcal{S}_\gamma^{\mathcal{T}},$$

and furthermore, if $\varkappa \neq \varkappa' \in \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+$, the sets $\varkappa \mathcal{S}_{\varkappa^{-1}\gamma}^\dagger$ and $\varkappa' \mathcal{S}_{(\varkappa')^{-1}\gamma}^\dagger$ are disjoint, and so we see that

$$\bigsqcup_{\varkappa \in \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+} \varkappa \mathcal{S}_{\varkappa^{-1}\gamma}^\dagger \subset \mathcal{S}_\gamma^{\mathcal{T}}.$$

The number of right $B_{\mathbb{M},f}^\dagger$-orbits in total on the left hand side is bounded by the number of $B_{\mathbb{M},f}^{\mathcal{T}}$-orbits on the right hand side multiplied by the maximum size of a fibre of the map

$$\mathbb{G}(\mathbb{Q})\mathbb{M}(\mathbb{A}_f)^\dagger / B_{\mathbb{M},f}^\dagger \to \mathbb{M}(\mathbb{A}_f)^{\mathcal{T}} / B_{\mathbb{M},f}^{\mathcal{T}}.$$

This is equal to the map

$$\mathbb{G}(\mathbb{Q})\mathbb{G}(\mathbb{A}_f)^+ / B_{\mathbb{G},f}^+ \to \mathbb{G}(\mathbb{A}_f) / B_{\mathbb{G},f}.$$

Suppose we are given two elements of the same fibre, that is, $\varkappa_1, \varkappa_2 \in \mathbb{G}(\mathbb{Q})$, and $m_1, m_2 \in \mathbb{G}(\mathbb{A}_f)^+$ such that

$$\varkappa_1 m_1 = \varkappa_2 m_2 b, \text{ for some } b \in B_{\mathbb{G},f}.$$

We have an injection (from the fact that $\mathbb{G}(\mathbb{A}_f)^+$ is the image of the simply connected cover, which is $B^{(1)}(\mathbb{A}_f)$, the reduced norm 1 units of B)

$$\mathrm{Nrd}: \mathbb{G}(\mathbb{A}_f)/\mathbb{G}(\mathbb{A}_f)^+ \to \mathbb{A}_{F,f}^\times / \left(\mathbb{A}_{f,F}^\times\right)^2.$$

(Note that while we have descended the group $G = PB^\times$ to $\mathbb{G}$ defined over $\mathbb{Q}$, the norm reduced map is still valued in F-adeles.) Since the image of $B_{\mathbb{G},f}$ is contained in $\mathcal{O}_{F,f}^\times \left(\mathbb{A}_{F,f}^\times\right)^2$, we see that the reduced norm of $\varkappa_2^{-1}\varkappa_1$ is an element of $F^\times / (F^\times)^2$ with even valuation at every finite place. Therefore the number of $B_{\mathbb{M},f}^\dagger$-cosets is bounded by the number of $\mathbb{G}(\mathbb{Q})^+$-cosets $\kappa_1\mathbb{G}(\mathbb{Q})^+$ with norm reduced having the same parity valuation at every finite place of F. This is in turn bounded by the cardinality of

$$\mathcal{E} := \left\{x \in F^\times : \nu_{\mathfrak{p}}(x) \in 2\mathbb{Z}, \forall \text{ primes } \mathfrak{p}\right\} / \left(F^\times\right)^2.$$

There is a exact sequence

$$1 \to \mathcal{O}_F^\times / \left(\mathcal{O}_F^\times\right)^2 \to \mathcal{E} \to \mathrm{Cl}_F[2] \to 1$$

given by taking $\alpha \in \mathcal{E}$ to the square root ideal of (α). Since F is totally real,

$$\#\left(\mathcal{O}_F^\times / \left(\mathcal{O}_F^\times\right)^2\right) = 2^{[F:\mathbb{Q}]}.$$

This implies the claimed result. □

Khayutin claims in Lemma 8.24 of [Kha19a] that for any $\gamma \in (\mathbb{G}\times\mathbb{G})(\mathbb{Q})$, we can use the contraction map $\mathbb{G}\times\mathbb{G} \to \mathbb{G}$ to deduce that $N_{[\gamma]}$ is equal to

$$\#\left(\mathrm{Ad}B_{\mathbb{T},f}^{\mathcal{T}} \setminus \mathrm{Ad}\mathcal{T}\mathrm{ctr}(\gamma) \cap \mathrm{ctr}(B_f')\right).$$

However this claim is not quite accurate, due to the following group theoretic lemma:

Lemma 6.1.10. *Let G be a group acting on a set X, and $X' \subset X$ be a subset. Suppose that $H \leq G$ is a subgroup such that $H \cdot X' \subset X'$. Define, for $x \in X$,*

$$\mathscr{S}_x = \left\{g \in G : g^{-1}\cdot x \in X'\right\} \subset G.$$

This subset is a union of right H-cosets. Suppose that $\mathscr{S}_x$ is finite. Then, there is a surjection

$$\mathscr{S}_x/H \to H \setminus (G\cdot x \cap X')$$

where the set on the right is the collection of H-orbits of $G\cdot x\cap X'$. However, this is not a bijection, and

$$|\mathscr{S}_x/H| = \sum_{[Hg^{-1}x]\in H\setminus(G\cdot x\cap X')} \left|\mathrm{Stab}_G(x)/\mathrm{Stab}_G(x)\cap gHg^{-1}\right|.$$

Proof. The surjectivity of the map, given by $g \mapsto g^{-1} \cdot x$ is obvious. The fibre above the point $Hg^{-1} \cdot x$ is given by

$$\mathrm{Stab}_G(x)gH/H \cong \mathrm{Stab}_G(x)/\mathrm{Stab}_G(x) \cap gHg^{-1}.$$

$\square$

The relevance of this lemma to the given situation is that the group $\mathbb{M}(\mathbb{A}_f)^{\mathcal{T}}$ is acting on $X = \mathbb{G}(\mathbb{A}_f) \times \mathbb{G}(\mathbb{A}_f)$ and $N_{[\gamma]} = |\mathscr{S}_\gamma / B^{\mathcal{T}}_{\mathbb{M},f}|$ in the notation of the lemma, where we have taken $X' = B'_f, H = B^{\mathcal{T}}_{\mathbb{M},f}$. However, Khayutin deduces that this is equal to the number of $B^{\mathcal{T}}_{\mathbb{M},f}$-orbits on the space $\mathbb{M}(\mathbb{A}_f)^{\mathcal{T}} \cdot \gamma \cap B'_f$ before translating this with the contraction map. Therefore a term is missing, and it is actually required to understand

$$\left|\mathbb{M}_\gamma(\mathbb{A}_f)^{\mathcal{T}} / \mathbb{M}_\gamma(\mathbb{A}_f)^{\mathcal{T}} \cap (l,t) B^{\mathcal{T}}_{\mathbb{M},f} (l,t)^{-1}\right|$$

for each $(l,t) \in \mathbb{M}(\mathbb{A}_f)^{\mathcal{T}}$ satisfying $l^{-1}\gamma t \in B'_f$. We now include this extra term in our result.

Proposition 6.1.11. *Suppose $\gamma \in (\mathbb{G} \times \mathbb{G})(\mathbb{Q})$, such that $\mathrm{ctr}(\gamma) \notin \mathbb{T}(\mathbb{Q})$. Then*

$$\sum_{\varkappa \in \mathbb{G}(\mathbb{Q})/\mathbb{G}(\mathbb{Q})^+} \mathrm{RO}^f_{\gamma,\varkappa}(B) \leq \Xi 2^{[F:\mathbb{Q}]} |\mathrm{Cl}_F[2]| m_{\mathbb{G}(\mathbb{A}_f)^+} \left(B^\dagger_{\mathbb{G},f}\right) m_{\mathcal{T}} \left(B^\dagger_{\mathbb{T},f}\right) M_{[\gamma]},$$

where for fixed $\mathbb{T}$, the constant Ξ depends only on whether $\mathbb{M}_\gamma$ is trivial or not,

$$\Xi = \begin{cases} 1, & \text{if } \mathbb{M}_\gamma = 1 \\ |\mathcal{T}[2]/\mathcal{T}[2] \cap gBg^{-1}|, & \text{if } \mathbb{M}_\gamma \cong \mathbb{T}[2], \end{cases}$$

and

$$M_{[\gamma]} := \# \left(\mathrm{Ad} B^{\mathcal{T}}_{\mathbb{T},f} \setminus \mathrm{Ad}\mathcal{T} \mathrm{ctr}(\gamma) \cap \mathrm{ctr}(B'_f)\right).$$

Proof. For $[\gamma] \in W_{\mathbb{Q}}$, since we have reduced to the case of compact stabilisers, $\mathrm{Stab}_{\mathbb{M}}(\gamma) = \mathbb{M}_\gamma$ is either trivial (when $\mathrm{ctr}(\gamma) \notin N_{\mathbb{G}}\mathbb{T}(\mathbb{Q})$) or $\mathbb{T}[2]$ (when $\mathrm{ctr}(\gamma) \in N_{\mathbb{G}}\mathbb{T}(\mathbb{Q}) \setminus \mathbb{T}(\mathbb{Q})$. In the trivial case, clearly the lemma above implies that

$$N_{[\gamma]} = \# \left(B^{\mathcal{T}}_{\mathbb{M},f} \setminus \left(\mathbb{M}(\mathbb{A}_f)^{\mathcal{T}} \cdot \gamma \cap B'_f\right)\right).$$

Now, the contraction map implies that

$$N_{[\gamma]} = \# \left(\mathrm{Ad} B^{\mathcal{T}}_{\mathbb{T},f} \setminus \mathrm{Ad}\mathcal{T} \mathrm{ctr}(\gamma) \cap \mathrm{ctr}(B'_f)\right).$$

In the other case, we need to compute

$$\left|\mathbb{M}_\gamma(\mathbb{A}_f) / \mathbb{M}_\gamma(\mathbb{A}_f) \cap (l,t) B^{\mathcal{T}}_{\mathbb{M},f} (l,t)^{-1}\right|$$

for $(l,t) \in \mathbb{G}(\mathbb{A}_f) \times \mathcal{T}$ satisfying $l^{-1}\gamma t \in B'_f$. There is a bijection

$$\begin{aligned} \mathbb{T}[2] &\longrightarrow \mathbb{M}_\gamma \\ \zeta &\longmapsto (\gamma_1\zeta^{-1}\gamma_1^{-1}, \zeta) \end{aligned}$$

and $\gamma_1\zeta^{-1}\gamma_1^{-1} = \gamma_2\zeta^{-1}\gamma_2^{-1}$. The condition that the image of ζ under this map lies in $(l,t)B^{\mathcal{T}}_{\mathbb{M},f}(l,t)^{-1}$ is equivalent to the two conditions

$$\zeta \in tgBg^{-1}t^{-1} \cap \mathcal{T}, \gamma_1\zeta^{-1}\gamma_1^{-1} \in l\xi_1 B\xi_1^{-1}l^{-1} \cap l\xi_2 B\xi_2^{-1}l^{-1}.$$

It is a simple calculation that both of these conditions are equivalent to $\zeta \in gBg^{-1} \cap \mathcal{T}$. Therefore

$$\left|\mathbb{M}_\gamma(\mathbb{A}_f)/\mathbb{M}_\gamma(\mathbb{A}_f) \cap (l,t)B^{\mathcal{T}}_{\mathbb{M},f}(l,t)^{-1}\right| = \left|\mathcal{T}[2]/\mathcal{T}[2] \cap gBg^{-1}\right|.$$

□

6.1.4 Sum over Multiplicative Functions

We continue the expansion in the joint CM setting by analysing the structure of the quotient $\mathrm{Ad}B^{\mathcal{T}}_{\mathbb{T},f} \setminus \mathrm{Ad}\mathcal{T}\mathbb{G}(\mathbb{Q})$ in a similar way to the method of [Kha17], however with some differences due to the potential non-triviality of the class group of F, the subtorus $\mathcal{T}$, as well as the infinitude of $\mathcal{O}_F^\times$. Our results differ slightly over $\mathbb{Q}$ with the results of Khayutin.

We restrict to the case that $B_{\mathbb{T},f} = \mathcal{K}_{\mathbb{T},f} = \mathbb{T}(\mathbb{A}_f) \cap g_f\mathcal{K}_f g_f^{-1}$. Note that in this case $B^{\mathcal{T}}_{\mathbb{T},f} = \mathcal{K}_{\mathcal{T}}$, as defined in Definition 2.5.1 of the volume of a toral homogeneous set. We also for convenience review the assumptions on $\mathcal{T}$:

1. $\mathcal{T} = \mathbb{T}(\mathbb{R})\prod_\nu \mathcal{T}_\nu$ splits as a product over finite places. In addition, we assume that $\mathcal{T}_\nu = \mathbb{T}(F_\nu)$ at all places ν where the quaternion algebra B is ramified.

2. $\mathcal{T}$ corresponds to a subgroup of the class group

$$\mathbb{T}(\mathbb{Q}) \setminus \mathbb{T}(\mathbb{A}_f)/\mathcal{K}_{\mathbb{T},f}$$

   and therefore that $\mathbb{T}(\mathbb{Q})\mathcal{K}_{\mathbb{T},f} \leq \mathcal{T}$. It seems possible to weaken this result to a subgroup of a ray class group (coming from a subgroup of $\mathcal{K}_{\mathbb{T},f}$, however for simplicity we stick to the full class group. In particular, this means that $B_{\mathbb{T},f} = B_{\mathcal{T}}$.

3. In addition, we assumed that $\mathbb{T}(\mathbb{A})^+ \leq \mathcal{T}$, so the quotient $\mathbb{T}(\mathbb{A}_f)/\mathcal{T}$ has exponent 2 (i.e. the subgroup of the class group corresponding to $\mathcal{T}$ contains the squares). This assumption was crucial in order to apply the measure theory, and seems difficult to remove.

4. Finally, we assume that $\mathbb{A}_F^\times \leq \mathcal{T}$, and that $\mathcal{T}$ is fixed by $\mathrm{Gal}(E/F)$. These assumptions are more for convenience and do not seem essential to the method.

Define $\mathbb{G}(\mathbb{A})_{\text{accessible}} = \mathrm{Ad}\mathbb{T}(\mathbb{A})\mathbb{G}(\mathbb{Q})$. From the representation of Proposition 2.4.1, we know that the elements $tgt^{-1} \in \mathbb{G}(\mathbb{A})_{\text{accessible}}$ have representations in $B(\mathbb{A})^\times$ (note that B denotes concurrently the quaternion algebra B and the subgroup $B_{\mathbb{T},f}$ - however no confusion should arise) given by

$$\left(\begin{pmatrix} a & \epsilon b \frac{\lambda_\nu}{^\sigma\lambda_\nu} \\ ^\sigma b \frac{^\sigma\lambda_\nu}{\lambda_\nu} & ^\sigma a \end{pmatrix}_\nu\right)_\nu \in B(\mathbb{A})^\times,$$

where $(\lambda_\nu)_\nu \in \mathbb{A}_E^\times$ is a lift of $t \in \mathbb{T}(\mathbb{A})$ and $a, b \in E$.

Definition 6.1.12. *Define the invariant function*

$$\mathrm{inv} : \mathrm{Ad}\mathcal{K}_{\mathbb{T},f}\mathrm{Ad}\mathbb{T}(\mathbb{R}) \setminus \mathrm{Ad}\mathbb{T}(\mathbb{A})\mathbb{G}(\mathbb{Q}) \to F^\times \setminus \left(E \times \left(E\mathbb{A}_{E,f}^{(1)}/\Lambda_f^{(1)}\right)\right)$$

as follows. Given an element $t\gamma t^{-1}$, with a representation as above in $B(\mathbb{A})^\times$, define

$$\mathrm{inv}(t\gamma t^{-1}) = F^\times \left(a, b\left(\frac{\lambda_\nu}{^\sigma\lambda_\nu}\right)\Lambda_f^{(1)}\right).$$

Since $\mathcal{K}_{\mathbb{T},f}$ is the image of $\Lambda_f^\times$, this is in fact invariant under $\mathrm{Ad}\mathcal{K}_{\mathbb{T},f}$. It is independent of the chosen representation of $t\gamma t^{-1}$ since any two of that form differ by $\mathbb{A}_F^\times \cap E = F^\times$.

Note that this is not the same definition of the invariant function as Khayutin gives in [Kha17, Definition 8.25], since over totally real fields when we have infinitely many units simply recording the ideal is not sufficient. However, some computations appear to be easier with this invariant function.

Proposition 6.1.13. *The fibres of the* inv *map have size 1 if $b = 0$. Otherwise the fibre has a faithful transitive action of the group*

$$\prod_{\nu \nmid \infty} H^1\left(\mathfrak{G}, \Lambda_\nu^\times\right),$$

where $\mathfrak{G} = \mathrm{Gal}(E/F)$.

Proof. First consider the fibre above $F^\times(x,0)$ for $x \in E^\times$. Then for any element of the fibre, $b=0$ and $F^\times a = F^\times x$ uniquely determines an element of $\mathbb{T}(\mathbb{Q})$, so the fibre has size 1.

Next, the fibre above $F^\times(a, b(\lambda_\nu^\sigma \lambda_\nu^{-1})_\nu \Lambda_f^{(1)})$ for $a \in E$, and $b \in E^\times, \lambda_\nu \in E_\nu^{(1)}$. The fibre consists of the images of

$$\left(\begin{pmatrix} a & \epsilon b \frac{\lambda_\nu}{{}^\sigma\lambda_\nu} v \\ {}^\sigma b \frac{{}^\sigma\lambda_\nu}{\lambda_\nu} v & {}^\sigma a \end{pmatrix}_\nu\right)_\nu,$$

where $v \in \Lambda_f^{(1)}$. The adjoint action of $\mathrm{Ad}\mathcal{K}_{\mathbb{T},f}$ acts on v by multiplication by elements of $\mathrm{cbd}\left(\Lambda_f^\times\right)$ where $cbd(x) = x/{}^\sigma x$. Therefore the size of the fibre is precisely

$$\left[\Lambda_f^{(1)} : \mathrm{cbd}\left(\Lambda_f^\times\right)\right] = \prod_{\nu\nmid\infty} \# H^1(\mathrm{Gal}(E/F), \Lambda_\nu^\times).$$

□

We now determine the image of the set $\mathrm{Ad}\mathbb{T}(\mathbb{A}_f)\mathrm{ctr}(\gamma) \cap \mathrm{ctr}(B'_f)$. First, let $\mathcal{C} = \{\mathfrak{c}_1, ..., \mathfrak{c}_h\}$ be a fixed set of representatives for the class group of F which we pick to have minimal norm amongst integral representatives of that class, and $\{\delta_\mathfrak{c} : \mathfrak{c} \in \mathcal{C}\} \subset \mathbb{A}_{F,f}^\times$ be a set of representatives for the ideals $\mathfrak{c} \in \mathcal{C}$. We also choose a fundamental domain $\mathscr{R} \subset (\mathbb{R}^{[F:\mathbb{Q}]})^{\mathrm{sum}=0}$ for the lattice $(\log \circ (|\cdot|_\nu)_{\nu|\infty})(\mathcal{O}_F^\times)$. Choose $\eta > 0$ such that there is a choice of fundamental domain $\mathscr{R}$ contained in $[-\eta,\eta]^{[F:\mathbb{Q}]}$. Let T be the number of units, $\zeta \in \mathcal{O}_F^\times$, mapping into $[2^{-4}e^{-2\eta}, 2^4 e^{2\eta}]^{[F:\mathbb{Q}]}$ under the map

$$\zeta \mapsto (|\zeta_\nu|_\nu : \nu|\infty).$$

This is a fixed constant depending only on F. Set

$$\rho = \left(N(\mathfrak{c})^2 N(\mathfrak{s})^{-1} \mathfrak{d}_f(x_f) \prod_{\nu|\infty} |\mathrm{Nm}(s_\nu)|_\nu \mathfrak{d}_\nu(x_\nu)\right)^{1/[F:\mathbb{Q}]}.$$

Definition 6.1.14. *Let $a \in A_S^+$ be the element used to construct the Bowen balls in Theorem 3.6.1, in particular it is regular at each place. Define*

$$\mathfrak{P} := \prod_{\substack{\nu|S \\ \alpha\in\Phi_\nu^+}} \mathfrak{p}_\nu^{v_\nu(\alpha(a_\nu))} \subset \mathcal{O}_F.$$

This ideal records the 'depth' of the Bowen ball $B^{(-n,n)}$. Notice that $2\log N\mathfrak{P}$ is equal to the entropy of the Haar measure of the action of a on $[\mathbb{G}(\mathbb{A})]$.

The following Proposition is crucial to constructing a sum over multiplicative functions. Notice that when $x = \mathrm{ctr}(\xi)$, the subset $\prod_\nu g_\nu B_\nu^{-1} x_\nu B_\nu g_\nu^{-1} s_\nu^{-1}$ of the Proposition is exactly $\mathrm{ctr}(B'_f)$, from Proposition 6.1.11. In short, this Proposition compares a rational lift with an optimal integral lift in order to produce an optimal rational lift. At the primes dividing S, we get a stronger congruence condition on this optimal lift when we restrict to the Bowen ball of level n.

Proposition 6.1.15. *Fix $(x_\nu)_\nu \in \mathbb{G}(\mathbb{A}_f)$ such that $\forall \nu \in S, x_\nu \in A_\nu$. Let $B_\nu = \Omega_\nu$ for $\nu \notin S$, and $B_\nu = \mathcal{K}_\nu^{(-n,n)} \subset \Omega_\nu$ for $\nu \in S$. If $h \in \mathbb{G}(\mathbb{A})_{\text{accessible}}$ is contained in $\prod_\nu g_\nu B_\nu^{-1} x_\nu B_\nu g_\nu^{-1} s_\nu^{-1}$ and*

$$\mathrm{inv}(h) = F^\times(a, b\Lambda_f^{(1)})$$

then for each $\mathfrak{c} \in \mathrm{Cl}_F$ there is at most T choices for the representatives $a \in E, b \in E\mathbb{A}_{E,f}^{(1)}/\Lambda_f^{(1)}$ satisfying

1. $a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}$
2. $b \in \left(\mathfrak{P}^n\right)^\sigma (\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}\widehat{\Lambda}_f$.
3. $\mathrm{Nm}(a) - \epsilon \mathrm{Nm}(b) \in F^\times$ *has valuations*

$$v_\nu(\mathrm{Nm}(a) - \epsilon\mathrm{Nm}(b)) = 2v_\nu(\mathfrak{c}) - v_\nu(\mathrm{Nm}(s_\nu)) + d(x_0, x_\nu \cdot x_0), \forall \nu \nmid \infty$$
$$\rho e^{-2\eta} \leq |\mathrm{Nm}(a) - \epsilon\mathrm{Nm}(b)|_\nu \leq \rho e^{2\eta} 2^8, \forall \nu | \infty.$$

4. *For every infinite place $\nu|\infty$,*

$$|a_\nu|_\nu \leq 2^8 |s_\nu|_\nu^{-1} e^\eta \rho.$$

The collection of $\mathfrak{c} \in \mathrm{Cl}_F$ where there is at least one such representative is contained inside a coset of $\mathrm{Cl}_F[2]$. Furthermore, if $h \in \mathrm{Ad}\mathcal{T}\mathbb{G}(\mathbb{Q})$ then $b \in E\mathcal{T}^{(1)}/\Lambda_f^{(1)}$ (where we consider $\mathcal{T} < \mathbb{A}_{E,f}^\times$).

Recall again that we denote the Galois action of E/F by ${}^\sigma(\cdot)$.

Proof. Let $(a, b\Lambda_f^{(1)})$ be a lift of $\mathrm{inv}(h)$ with $a \in F^\times$, and $b \in E\mathbb{A}_{E,f}^{(1)}/\Lambda_f^{(1)}$.

For each place ν of F, choose (via Proposition 2.4.5) a representative $r_\nu \in B^\times(F_\nu) \cap \mathcal{O}_\nu$ of $g_\nu^{-1} h_\nu s_\nu g_\nu \in \Omega_\nu x_\nu \Omega_\nu$ satisfying

$$v_\nu\left(\mathrm{Nrd}(r_\nu)\right) = d(x_0, x_\nu \cdot x_0), \forall \nu \nmid \infty$$
$$2^{-8} \leq |\mathrm{Nrd}(r_\nu)|_\nu \, \mathfrak{d}_\nu(x_\nu) \leq 1, \forall \nu | \infty.$$

For $\nu|S$, we can additionally stipulate $r_\nu \in \mathcal{O}_\nu^{(-n,n)}$. Due to Proposition 2.4.2, there exists $(\alpha_\nu)_{\nu\nmid\infty}, (\beta_\nu)_{\nu\nmid\infty} \in \widehat{\Lambda}_f$, the $\mathcal{O}_{F,f}$-dual of Λ_f, and $\mu \in \mathbb{A}_F^\times$ such that at finite places $\nu \nmid \infty$

$$\mu_\nu \begin{pmatrix} \alpha_\nu s_\nu^{-1} & \beta_\nu {}^\sigma s_\nu^{-1} \upsilon_\nu \tau_\nu \\ {}^\sigma\beta_\nu s_\nu^{-1}/\tau_\nu & {}^\sigma\alpha_\nu^\sigma s_\nu^{-1} \end{pmatrix} = \begin{pmatrix} a & \epsilon b \\ {}^\sigma b & {}^\sigma a \end{pmatrix}$$

and for the split infinite places (where $\epsilon_\nu = {}^\sigma f_\nu f_\nu$ for some $f \in E^\times$), there are $\alpha_\nu, \beta_\nu \in E_\nu$ such that $|\mathrm{Nm}(\alpha_\nu)|_\nu + |\mathrm{Nm}(\beta_\nu)| \leq 1$ and

$$\mu_\nu \begin{pmatrix} \alpha_\nu s_\nu^{-1} & \beta_\nu {}^\sigma s_\nu^{-1} f_\nu \\ {}^\sigma\beta_\nu s_\nu^{-1}/f_\nu & {}^\sigma\alpha_\nu^\sigma s_\nu^{-1} \end{pmatrix} = \begin{pmatrix} a & \epsilon b \\ {}^\sigma b & {}^\sigma a \end{pmatrix}$$

and for the ramified infinite places there are $\alpha_\nu, \beta_\nu \in E_\nu$ such that $|\mathrm{Nm}(\alpha_\nu) - \epsilon\mathrm{Nm}(\beta_\nu)|_\nu \leq 1$, and

$$\mu_\nu \begin{pmatrix} \alpha_\nu s_\nu^{-1} & \epsilon\, {}^\sigma s_\nu^{-1} \beta_\nu \\ {}^\sigma\beta_\nu s_\nu^{-1} & {}^\sigma\alpha_\nu\, {}^\sigma s_\nu^{-1} \end{pmatrix} = \begin{pmatrix} a & \epsilon b \\ {}^\sigma b & {}^\sigma a \end{pmatrix}.$$

The conditions on $\mathrm{Nrd}(r_\nu)$ correspond to the additional statements

$$\begin{aligned} v_\nu(\mathrm{Nm}(\alpha_\nu) - \upsilon_\nu \mathrm{Nm}(\beta_\nu)) &= d(x_0, x_\nu \cdot x_0), \forall \nu \nmid \infty \\ 2^{-8} \leq |\mathrm{Nm}(\alpha_\nu) - \upsilon_\nu \mathrm{Nm}(\beta_\nu)|_\nu \mathfrak{d}_\nu(x_\nu) &\leq 1, \nu|\infty \text{ split} \\ 2^{-8} \leq |\mathrm{Nm}(\alpha_\nu) - \epsilon\mathrm{Nm}(\beta_\nu)|_\nu &\leq 1, \nu|\infty \text{ ramified.} \end{aligned}$$

By altering $(\alpha_\nu, \beta_\nu)_{\nu\nmid\infty}$ by a common factor of $\mathcal{O}_{F,f}^\times$ and (a, b) by a common factor of $F^\times$, we can assume $(\mu_\nu)_\nu \in \delta_\mathfrak{c} F_\infty^\times$ for some $\mathfrak{c} \in \mathcal{C}$. Any lift $(a, b\Lambda_f^{(1)})$ satisfying this additional condition must vary by a factor of $\mathcal{O}_F^\times$ since

$$v_\nu(\mathrm{Nm}(a) - \epsilon\mathrm{Nm}(b)) = v_\nu(\mathfrak{c}^2 \mathrm{Nm}\mathfrak{s}^{-1}) + d(x_0, x_\nu \cdot x_0)$$

is completely determined for fixed $\mathfrak{c}$. It also shows that the possible values of $\mathfrak{c}$ which can arise are (contained in) a coset of $\mathrm{Cl}_F[2]$. Now, we fix one possible $\mathfrak{c}$.

By the product formula for F, we know that

$$\prod_{\nu|\infty} |\mathrm{Nm}(a) - \epsilon\mathrm{Nm}(b)|_\nu = N(\mathfrak{c}^2) N(\mathfrak{s}^{-1}) \mathfrak{d}_f(x_f).$$

This implies that

$$N(\mathfrak{c}^2)N(\mathfrak{s}^{-1})\mathfrak{d}_f(x_f) \prod_{\nu|\infty} |\mathrm{Nm}(s_\nu)|_\nu \mathfrak{d}_\nu(x_\nu) \leq \prod_{\nu|\infty} |\mu_\nu|_\nu^2 \leq N(\mathfrak{c}^2)N(\mathfrak{s}^{-1})\mathfrak{d}_f(x_f) \prod_{\nu|\infty} 2^8 |\mathrm{Nm}(s_\nu)|_\nu \mathfrak{d}_\nu(x_\nu).$$

We have assumed that fundamental domain $\mathscr{R}$ is contained in $[-\eta, \eta]^{[F:\mathbb{Q}]}$. This means that whatever the value of $\prod_{\nu|\infty} |\mu_\nu|_\nu$, we can adjust by an element of $\mathcal{O}_F^\times$ to ensure that

$$e^{-\eta} \sqrt[d]{\prod_{\nu|\infty} |\mu_\nu|_\nu} \leq |\mu_\nu|_\nu \leq e^{\eta} \sqrt[d]{\prod_{\nu|\infty} |\mu_\nu|_\nu}, \forall \nu|\infty.$$

The uncertainty in the product by $2^{4[F:\mathbb{Q}]}$ means that any two values of $\mathcal{O}_F^\times$ moving $|\mu_\nu|_\nu$ into this range must differ by a unit $\zeta \in \mathcal{O}_F^\times$ satisfying

$$2^{-4}e^{-2\eta} \leq |\zeta|_\nu \leq 2^4 e^{2\eta}, \forall \nu | \infty.$$

Therefore, we get at most T lifts $(a, b\Lambda_f^{(1)})$ for the given value of $\mathfrak{c}$. For $(\mu_\nu)_\nu$ satisfying these conditions, we get

$$\rho e^{-2\eta} \leq |\mathrm{Nm}(a) - \epsilon \mathrm{Nm}(b)|_\nu \leq \rho e^{2\eta} 2^8.$$

We now see that for any of these representatives,

$$|a_\nu|_\nu = |\mu_\nu|_\nu |\alpha_\nu|_\nu |s_\nu|_\nu^{-1}, \forall \nu | \infty.$$

By the fact that $|\alpha_\nu|_\nu \leq 1$, we get

$$|a_\nu|_\nu \leq 2^8 |s_\nu|_\nu^{-1} e^\eta \rho.$$

The extra conditions on b at the places dividing S follows exactly as in the rational case, by using the stronger result of Proposition 2.4.5 under the assumption that $x_\nu \in A_\nu$ and $h \in \mathcal{K}_\nu^{(-n,n)} x_\nu \mathcal{K}_\nu^{(-n,n)}$. The image of $\mathrm{Ad}\mathcal{T}\mathbb{G}(\mathbb{Q})$ is clear. □

We now move to a sum over ideals using these small representatives.

Definition 6.1.16. *Let*

$$\mathscr{I}(\Lambda) = \mathbb{A}_{E,f}^\times / \Lambda_f^\times$$

which is the set of proper invertible Λ-ideals. Let $\mathscr{I}(\Lambda)^{pg}$ be the image of $E^\times \mathbb{A}_{E,f}(1)$ in $\mathscr{I}(\Lambda)$, which is all of the ideals whose class lies in the principal genus. Finally, let $\mathscr{I}(\Lambda)_0^{pg} = \mathscr{I}(\Lambda)^{pg} \cup \{0\}$. We also need to be able to record the subgroup $\mathcal{T}$. For this, we define $\mathscr{I}(\Lambda)^{pg,\mathcal{T}}$ to be the image in $\mathscr{I}(\Lambda)$ of $E\mathcal{T}^{(1)}$, and $\mathscr{I}(\Lambda)_0^{pg,\mathcal{T}} = \mathscr{I}(\Lambda)^{pg,\mathcal{T}} \cup \{0\}$.

The norm of an element $b = b_0(\mu_\nu)_\nu \in E^\times \mathbb{A}_{E,f}^{(1)}$ with $b_0 \in E^\times$ is everywhere equal to $\mathrm{Nm}(b_0) \in F^\times$. Define $\mathrm{Nm}(b) := \mathrm{Nm}(b_0) \in F^\times$. To an ideal $\mathfrak{b} \in \mathscr{I}(\Lambda)_0^{pg}$ we associate a norm $\mathrm{Nm}(\mathfrak{b}) \in F^\times/\mathcal{O}_F^\times$ by choosing a representative in $E\mathbb{A}_{E,f}^{(1)}$. In other words, we associate a principal ideal in F.

Note that under the assumptions we have made on $\mathcal{T}$, the subgroup $\mathcal{T}^{(1)}$ of norm one elements is precisely the image of $t \mapsto t/{}^\sigma t$ for $t \in \mathcal{T}$. An element $x \in \mathbb{A}_{E,f}^{(1)}$ is in $\mathcal{T}^{(1)}$ if and only if $\chi(x) = 1$ for all $\chi \in \mathcal{T}^\perp$.

To a pair of representatives $(a, b\Lambda_f^{(1)})$ satisfying the properties of Proposition 6.1.15, we associate

$$(a, \mathfrak{b}) := (a, b\Lambda_f^\times) \in E \times \mathscr{J}(\Lambda)_0^{pg}.$$

In other words, we take the Λ-ideal $\mathfrak{b}$ generated by $b\Lambda_f^{(1)}$. The point of going via Proposition 6.1.15 is that while the map

$$E\mathbb{A}_{E,f}^{(1)}/\Lambda_f^{(1)} \to \mathbb{A}_{E,f}^\times/\Lambda_f^\times \cup \{0\}$$

has infinite fibres, this is no longer the case if we restrict to the subset of the domain where the conditions for $b\Lambda_f^{(1)}$ in Proposition 6.1.15 hold. Therefore, we can pass to a sum over ideals bypassing the issue of infinitely many units.

Definition 6.1.17. *Let $\mathscr{M}$ be the finite set of $\zeta \in F^\times$ satisfying*

$$v_\nu(\zeta) = 2v_\nu(\mathfrak{c}) - v_\nu(\mathrm{Nm}(s_\nu)) + d(x_0, x_\nu \cdot x_0), \forall \nu \nmid \infty$$
$$\rho e^{-2\eta} \leq |\zeta|_\nu \leq \rho e^{2\eta} 2^8, \forall \nu | \infty.$$

The size of $\mathscr{M}$ is bounded uniformly depending only on F.

Lemma 6.1.18. *Let $\mathscr{S}_\mathfrak{c} \subset E \times E\mathbb{A}_{E,f}^{(1)}/\Lambda_f^{(1)}$ be subset satisfying the conditions of Proposition 6.1.15, i.e. with 'small' representatives, for the class $[\mathfrak{c}] \in \mathrm{Cl}_F$. Then the map*

$$\mathscr{S}_\mathfrak{c} \to E \times \mathscr{J}(\Lambda)_0^{pg}$$

has fibres of size at most T. The image is contained in the subset satisfying

1. *$a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}$,*

2. *For every infinite place $\nu|\infty$,*

$$|a_\nu|_\nu \leq 2^8 |s_\nu|_\nu^{-1} e^\eta \rho$$

3. *$\mathfrak{b} \subset (\mathfrak{P}^n)^\sigma (\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}\widehat{\Lambda}$,*

4. *$\mathrm{Nm}(\mathfrak{b}) = (\frac{1}{\epsilon}(\mathrm{Nm}(a) - \zeta))$ for some $\zeta \in \mathscr{M}$.*

Proof. Consider $(a, b_0(\mu_\nu)_\nu\Lambda_f^\times)$ in the same image, with $b_0 \in E$, $(\mu_\nu)_\nu \in \mathbb{A}_{E,f}^{(1)}$. Clearly, if $b_0 = 0$, the fibre has size 1.

Otherwise, if $(a, b_0'(\mu_\nu')_\nu\Lambda_f^{(1)})$ is in the fibre, $b_0'\mu_\nu' = \lambda_\nu b_0 \mu_\nu$ for some $\lambda_\nu \in \Lambda_\nu^\times$. There are at most T possibilities for $\mathrm{Nm}(b') = \mathrm{Nm}(\lambda_\nu)\mathrm{Nm}(b_0)$. Suppose $\mathrm{Nm}(b) = \mathrm{Nm}(b')$, then $\lambda_\nu \in \Lambda_\nu^{(1)}$, and these points in the fibre are the same. ☐

We also note that as in the rational case, we can rule out part of the fibres in Proposition 6.1.13 using the explicit identification of the non-trivial elements of the local cohomology groups as in Proposition A.3.

Proposition 6.1.19. *Let $h \in \mathbb{G}(\mathbb{A})_{accessible}$ have small representative $(a, b\Lambda_f^{(1)})$ as described in Proposition 6.1.15. For primes $\nu \nmid \mathfrak{d}_{\Lambda/\mathcal{O}_F}$, the group $H^1(\mathfrak{G}, \Lambda_\nu^\times)$ is trivial. For $\nu | \mathfrak{d}_{\Lambda/\mathcal{O}_F}$ coprime to 2 where B_ν is split, and $b \in {}^\sigma(\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}\Lambda$ (in particular notice Λ, rather than $\widehat{\Lambda}$), not both elements of $H^1(\mathfrak{G}, \Lambda_\nu^\times) \cdot h_\nu$ can lie in $g_\nu B_\nu^{-1} x_\nu B_\nu g_\nu^{-1} s_\nu$.*

The proof is identical to Proposition 8.32 of [Kha17].

Definition 6.1.20. *Define the function*

$$f : F^\times \to \mathbb{N}$$

such that $f(x)$ is equal to the number of ideals $\mathfrak{b} \subset \mathfrak{P}^{n\sigma}(\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}\widehat{\Lambda}$ for which $\mathrm{Nm}(\mathfrak{b}) = (x)$ and $\mathfrak{b} \in \mathscr{J}(\Lambda)^{pg,\mathcal{T}}$. This clearly descends to a function on principal ideals.

Also, define the multiplicative function $r : F^\times \to \mathbb{N}$ by the restriction to principal ideals of a multiplicative function on all fractional ideals of F (which we also call r) requiring that

$$r(\mathfrak{p}^k) = \begin{cases} 1, & \text{if } \mathfrak{p} \nmid D_{\Lambda/F}, \text{ else} \\ 2, & \text{if } G \text{ is not split at } \mathfrak{p}, \text{ else} \\ 2^{[F_\nu:\mathbb{Q}_2]}, & \text{if } \mathfrak{p}|2, \text{ else} \\ 1, & \text{if } k < \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}^2(\mathrm{Nm}(\mathfrak{s}\mathfrak{t}))^{-1}), \\ 2, & o/w \end{cases}$$

Theorem 6.1.21. *Suppose that Assumption 6.1.5 holds. Then the correlations can be bounded as*

$$\mathrm{Cor}[\mu, \nu](B^{(-n,n)}) \ll_F \mathrm{vol}([\mathcal{T}g])^{-1}\mathrm{vol}\left([\mathbb{G}^\Delta(\mathbb{A})^+\xi]\right)^{-1} e^{-2nh_{\mathbb{G}^{sc}}(a)}(S_1 + S_2)$$

Where $h_{\mathbb{G}^{sc}}(a)$ is the entropy of the Haar measure on $\mathbb{G}^{sc}$ under the action of a, and

$$S_1 := 2^{\omega_F(D_{\Lambda/F})} \sum_{[\mathfrak{c}] \in \mathrm{Cl}_F} \#\left\{a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda} : |a_\nu|_\nu \le 2^8 |s_\nu|_\nu^{-1} e^\eta \rho\right\}$$

$$S_2 := \sum_{\substack{[\mathfrak{c}] \in \mathrm{Cl}_F \\ \zeta \in \mathscr{M}}} \sum_{\substack{a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda} \\ |a_\nu|_\nu \le 2^8|s_\nu|_\nu^{-1} e^\eta \rho, \forall \nu | \infty}} f\left(\frac{\mathrm{Nm}(a) - \zeta}{\epsilon}\right) r\left(\frac{\mathrm{Nm}(a) - \zeta}{\epsilon}\right).$$

where ω_F is the prime counting function on ideals of F.

This is the generalisation of Theorem 8.7 of [Kha17] - note that as mentioned after Lemma 6.1.6, Assumption 6.1.5 will hold for tori $\mathbb{T}_i$ with $i \gg 1$ under the assumptions of our main theorem. Note, however, that the totally real case does not allow a sum over a product of multiplicative functions, f, g as in that Theorem, since we have infinitely many units. Instead in Theorem 6.1.21 we have first a sum over integral elements (which replaces the function g in Khayutin's result), then a count of ideals. Let us briefly recall the dependence of each constant here on the conditions:

- η, T depend only on the field F,
- ϵ depends only on the quaternion algebra B.
- $\mathfrak{s}$ is the ideal generated by the twist $s \in \mathbb{T}(\mathbb{A})$,
- ρ depends on $\mathfrak{c}, \mathfrak{s}, \xi$.
- Λ depends on the embedding $\mathbb{T} \to \mathbb{G}$.
- The size of $\mathscr{M}$ is bounded uniformly (depending only on F), however $\mathscr{M}$ depends on $\mathfrak{c}, \mathfrak{s}, \xi$.

Proof. From Corollary 6.1.7 (hence the need for Assumption 6.1.5), Lemma 6.1.8 and Proposition 6.1.11, we see that

$$\begin{aligned}\operatorname{Cor}[\mu,\nu] \leq 2^{[F:\mathbb{Q}]}|\mathrm{Cl}_F[2]| m_{\mathbb{G}(\mathbb{A}_f)^+}(B^{\dagger}_{\mathbb{G},f}) m_{\mathcal{T}}(B^{\dagger}_{\mathbb{T},f}) m_{\mathbb{T}(\mathbb{R})}(\mathbb{T}(\mathbb{R}))\cdot \\ m_{\mathbb{G}(\mathbb{R})^+}\left(\xi_{1,\infty}\Omega^2_{\infty}\xi^{-1}_{1,\infty} \cap \xi_{2,\infty}\Omega^2_{\infty}\xi^{-1}_{2,\infty}\right)\cdot \\ \left(\frac{\Xi}{2}\sum_{\substack{[\gamma]\in W_{\mathbb{Q}}\\ \operatorname{ctr}(\gamma)\in w_{\mathbb{T}(\mathbb{Q})}}} M_{[\gamma]} + \sum_{\substack{[\gamma]\in W_{\mathbb{Q}}\\ \operatorname{ctr}(\gamma)\notin N_{\mathbb{G}}\mathbb{T}(\mathbb{Q})}} M_{[\gamma]}\right)\end{aligned}$$

The volume computation to deal with homogeneous Hecke sets is exactly as in the rational case, and for the toral set we use

$$m_{\mathcal{T}}(B^{\dagger}_{\mathbb{T},f}) m_{\mathbb{T}(\mathbb{R})}(\mathbb{T}(\mathbb{R})) = \operatorname{vol}\left([\mathcal{T}g]\right)^{-1}.$$

By applying the invariant function map, and choosing small representatives, noting that the function r defined above gives an upper bound on the fibre (as shown in Appendix A), we find that

$$\sum_{\substack{[\gamma]\in W_{\mathbb{Q}}\\ \operatorname{ctr}(\gamma)\in w_{\mathbb{T}}\mathbb{T}(\mathbb{Q})}} M_{[\gamma]} \leq \sum_{[\mathfrak{c}]\in\mathrm{Cl}_F} \#\left\{a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda} : |a_\nu|_\nu \leq 2^8 |s_\nu|_\nu^{-1} e^{\eta}\rho\right\}.$$

For the second, more important, term we get

$$\sum_{\substack{[\gamma]\in W_{\mathbb{Q}}\\ \mathrm{ctr}(\gamma)\notin N_{\mathbb{G}}\mathbb{T}(\mathbb{Q})}} M_{[\gamma]} \le T \sum_{\substack{[\mathfrak{c}]\in \mathrm{Cl}_F\\ \zeta\in\mathscr{M}}} \sum_{\substack{a\in\mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}\\ |a_\nu|_\nu\le 2^8|s_\nu|_\nu^{-1}e^\eta\rho}} f\left(\frac{\mathrm{Nm}(a)-\zeta}{\epsilon}\right) r\left(\frac{\mathrm{Nm}(a)-\zeta}{\epsilon}\right)$$

Finally, we compute Ξ given our data. It only appears for the S_1 term since it is the size of $\mathbb{M}_\gamma(\mathbb{A}_f)/\mathbb{M}_\gamma(\mathbb{A}_f)\cap B_{\mathbb{M},f}$. This is equal to

$$\mathcal{T}[2]/\mathcal{T}[2]\cap B_{\mathbb{T},f}.$$

This can be computed locally, and in fact we see that a place (coprime to 2) contributes a factor of 2 precisely if no element of $\Lambda_\nu^\times$ has trace zero. If $\Lambda_\nu=\mathcal{O}_{E_\nu}$ then such elements exist when ν is unramified in E. If $\Lambda_\nu=\mathcal{O}_{F_\nu}+\mathfrak{f}_\nu\mathcal{O}_{E_\nu}$ then for $\nu\nmid 2$ and $x\in\mathcal{O}_{F_\nu}^\times$,

$$\mathrm{Tr}(x+f_\nu y)=2x+y\mathrm{Tr}f_\nu\in\mathcal{O}_{F_\nu}^\times$$

is non-zero. Thus we get that $\Xi\le 2^{[F:\mathbb{Q}]}2^{\omega_F(D_{\Lambda/F})}$. □

The S_1-term in the Theorem above is easy to compute simply via the geometry of numbers. In particular, with F fixed, we get

$$S_1\asymp 2^{\omega_F(D_{\Lambda/F})}\mathfrak{d}_f(\mathrm{ctr}(\xi_f))\mathfrak{d}_\infty(\mathrm{ctr}(\xi_\infty))\frac{N\mathfrak{f}_\Lambda}{\sqrt{\Delta_E}}.$$

The difficult term is S_2. However, since the size of $\mathscr{M}$ is uniformly bounded, it suffices to bound each inner sum individually. That is, we wish to bound

$$\sum_{\substack{a\in\mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}\\ |a_\nu|_\nu\le 2^8|s_\nu|_\nu^{-1}e^\eta\rho}} f\left(\frac{\mathrm{Nm}(a)-\zeta}{\epsilon}\right) r\left(\frac{\mathrm{Nm}(a)-\zeta}{\epsilon}\right).$$

Proposition 6.1.22. *The sum above is bounded above by*

$$2^{[F:\mathbb{Q}]}\sum_{\substack{a\in\mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}\\ |a_\nu|_\nu\le 2^8|s_\nu|_\nu^{-1}e^\eta\rho\\ \upsilon\mathfrak{P}^{2n}|(\mathrm{Nm}(a)-\zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2}D_{\Lambda/F}}}(f_0\cdot r_0)\left(\frac{(\mathrm{Nm}(a)-\zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2}D_{\Lambda/F}}{\upsilon\mathfrak{P}^{2n}}\right),$$

where f_0, r_0 are multiplicative functions defined on $\mathbb{A}_{F,f}^\times/\mathrm{Nm}\Lambda_f^\times$ by

$$f_0(x_\nu\mathrm{Nm}\Lambda_\nu^\times)=\#\left\{\mathfrak{b}\subset\Lambda_\nu:\mathrm{Nm}\mathfrak{b}=x_\nu\mathrm{Nm}\Lambda_\nu^\times,[\mathfrak{b}]\in\mathcal{T}\right\}$$

and

$$r_0(x_\nu\mathrm{Nm}\Lambda_\nu^\times)=\begin{cases}2, & \textit{if } \mathrm{ord}_\nu(x_\nu)\ge\mathrm{ord}_\nu D_{\Lambda/F},\nu\nmid 2\\ 1, & o/w.\end{cases}$$

Proof. The conditions for $\mathfrak{b}$ which are counted by f are $\mathfrak{b} \subset \mathfrak{P}^{n\sigma}(\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}\widehat{\Lambda}$, $\mathrm{Nm}(\mathfrak{b}) = (x)$, $\mathfrak{b} \in \mathscr{J}(\Lambda)^{pg}$ and $[\mathfrak{b}] \in \mathcal{T}$. For $a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}$ to satisfy $f\left(\frac{1}{\epsilon}(\mathfrak{N}\mathfrak{m}(a) - \zeta)\right) \neq 0$, we require that

$$\left(\frac{\mathrm{Nm}(a) - \zeta}{\epsilon}\right) \subset \mathfrak{P}^{2n}\mathrm{Nm}(\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}^2 D_{\Lambda/F}^{-1}$$

which, using the fact that $\epsilon = \upsilon_\nu \tau_\nu^\sigma \tau_\nu$, we see is equivalent to

$$\upsilon\mathfrak{P}^{2n}|(\mathrm{Nm}(a) - \zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2}D_{\Lambda/F}.$$

Therefore, if we let $\mathfrak{b} = \mathfrak{P}^{n\sigma}(\mathfrak{s}\mathfrak{t})^{-1}\mathfrak{c}\mathfrak{b}_0\widehat{\Lambda}$, then $\mathfrak{b}$ contributes to $f(\frac{1}{\epsilon}(\mathrm{Nm}(a) - \zeta))$ iff $\mathfrak{b}_0$ contributes to $f_0(\frac{1}{\upsilon\mathfrak{P}^{2n}}(\mathrm{Nm}(a) - \zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2}D_{\Lambda/F})$, $\mathfrak{b}_0 \in \mathcal{T}$ (this uses our assumptions on $\mathcal{T}$ along with the fact that at primes ν unramified in B, $\tau_\nu = \mathfrak{d}_{\Lambda_\nu/F}$) *and* $[\mathfrak{b}_0] = [\mathfrak{P}^{-n\sigma}(\mathfrak{s}\mathfrak{t})\mathfrak{c}^{-1}\mathfrak{d}_{\Lambda/F}]\mathrm{Pic}(\Lambda)^{pg}$. However, from principal genus theory, we know that we can detect the principal genus by testing $\mathrm{Nm}\mathfrak{b}_0 \in \mathbb{A}_{F,f}^\times/\mathrm{Nm}\Lambda_f^\times$ against all the characters of

$$F^\times \setminus \mathbb{A}_F^\times/F_\infty^{\gg 0}\mathrm{Nm}\Lambda_f^\times.$$

If $\mathrm{Nm}\mathfrak{b}_0 = \frac{1}{\upsilon\mathfrak{P}^{2n}}(\mathrm{Nm}(a) - \zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2}D_{\Lambda/F})$, then all we require is that for all such characters χ which need testing,

$$\chi\left(\frac{1}{\upsilon\mathfrak{P}^{2n}}(\mathrm{Nm}(a) - \zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2}D_{\Lambda/F}\right) = \chi\left(\mathfrak{P}^{-2n}\mathrm{Nm}(\mathfrak{s}\mathfrak{t})\mathfrak{c}^{-2}D_{\Lambda/F}\right)$$

This is equivalent to

$$\chi((\mathrm{Nm}(a) - \zeta)\epsilon^{-1}) = 1$$

which is immediate since χ vanishes on $F^\times$. All that is left is to deal with r, however it is clear that under this translation from $\mathfrak{b}$ to $\mathfrak{b}_0$, r_0 corresponds to r except we have assumed all places above 2 contribute to r, and this is where the factor of $2^{[F:\mathbb{Q}]}$ enters. □

Note that, in order to conclude the proof of Theorem 2.6.5, we need to prove that as $i \to \infty$, the sum of Proposition 6.1.22 is bounded above (uniformly for $\mathrm{ctr}(\xi)$ varying over a compact subset of $\mathbb{G}(\mathbb{A})$) by

$$\mathrm{vol}([\mathcal{T}g])e^{-\epsilon n h_{\mathbb{G}^{sc}}(a)}$$

for some $\epsilon > 0$. Recall from Definition 6.1.14 that $e^{h_{\mathbb{G}^{sc}}(a)} = N\mathfrak{P}$. The presence of ξ in Proposition 6.1.22 is via ρ.

6.2 The Spectral Expansion

Just for interest, we also record here alternative approach to an expansion of the correlation, but not one that we will use now - we would however be interested to see whether a fully automorphic approach to this correlation could give any meaningful results. We could consider a spectral expansion of the correlation in terms of the L^2-eigenfunctions on the space $[\mathbb{G}(\mathbb{A})]$. This works as follows.

Let's just assume for now that everything is already simply connected. We are comparing $[\mathbb{T}(\mathbb{A})g]$ with $[\mathbb{L}(\mathbb{A})h]$. Then, we get that

$$\begin{aligned}
\mathrm{Corr}(\mu,\nu)[f] &= \int_{[\mathbb{T}(\mathbb{A})]}\int_{[\mathbb{L}(\mathbb{A})]}\sum_{\gamma\in\mathbb{G}(\mathbb{Q})} f(g^{-1}x^{-1}\gamma y h)dydx \\
&= \int\int\sum_{\phi}\phi(x)\left(\int_{[\mathbb{G}(\mathbb{A})]}\overline{\phi(z)}\sum_{\gamma\in\mathbb{G}(\mathbb{Q})} f(g^{-1}z^{-1}\gamma y h)dz\right)dydx \\
&= \int\int\sum_{\phi}\phi(x)\left(\int_{\mathbb{G}(\mathbb{A})}\overline{\phi(z)}f(g^{-1}z^{-1}yh)dz\right)dydx \\
&= \sum_{\phi}\left(\int_{[\mathbb{T}(\mathbb{A})]}\phi(x)dx\right)\left(\int_{[\mathbb{L}(\mathbb{A})]}((Rf')\overline{\phi})(y)dy\right)
\end{aligned}$$

where,

$$((Rf')\overline{\phi})(y) = \int_{\mathbb{G}(\mathbb{A})}\overline{\phi(z)}f'(y^{-1}z)dz = \int_{\mathbb{G}(\mathbb{A})} f'(z)\overline{\phi(yz)}dz$$

and $f'(z) = f(g^{-1}z^{-1}h)$. Thus, we only need to pick out the forms ϕ for which the torus period is non-zero.

Alternatively, and completely symmetrically, we could consider the spectral expansion in the variable y, and then we would obtain the relation

$$\mathrm{Corr}(\mu,\nu)[f] = \sum_{\phi}\left(\int_{[\mathbb{L}(\mathbb{A})]}\phi(y)dy\right)\left(\int_{\mathbb{T}(\mathbb{A})}((Rf'')\overline{\phi})(x)dx\right)$$

where $f''(z) = f(g^{-1}zh) = f'(z^{-1})$.

Furthermore, we could actually do both expansions, and we get

$$\begin{aligned}
\mathrm{Corr}\left(\mu,\nu\right)[f] &= \sum_{\phi,\varphi}\left(\int_{[\mathbb{T}(\mathbb{A})]}\phi(x)dx\right)\left(\int_{[\mathbb{L}(\mathbb{A})]}\varphi(y)dy\right)\int_{z\in\mathbb{G}(\mathbb{A})} f'(z)\left(\int_{[\mathbb{G}(\mathbb{A})]}\overline{\varphi(w)\phi(wz)}dw\right)dz \\
&= \sum_{\phi,\varphi}\mathcal{P}_{\mathbb{T}}(\phi)\mathcal{P}_{\mathbb{L}}(\varphi)a_{\varphi}\left(R(f')\overline{\phi}\right).
\end{aligned}$$

Here, $a_\phi(\cdot)$ refers to the coefficient of ϕ in the L^2-decomposition of an element of $L^2([\mathbb{G}(\mathbb{A})])$ into our chosen basis.

6.2.1 The Joint Quaternion Algebra Case

Let's consider doing this in the case where $\mathbb{T} = \mathbb{T}^{\Delta}, \mathbb{L} = \mathbb{G}^{\Delta} \leq \mathbb{G} \times \mathbb{G}$ for $\mathbb{G}$ the projective group attached to a definite quaternion algebra. Then the forms ϕ are of the form $\phi_1 \otimes \phi_2$ for a pair of forms ϕ_1, ϕ_2 for $\mathbb{G}$. Then, we get from the second expansion that

$$\mathrm{Corr}(\mu, \nu)[f] = \sum_{\phi_1, \phi_2} \left(\int_{[\mathbb{G}(\mathbb{A})]} \phi_1(y)\phi_2(y)dy \right) \left(\int_{[\mathbb{T}(\mathbb{A})^{\Delta}]} ((Rf'')\overline{\phi_1 \otimes \phi_2}(x)dx \right)$$

The first integral is non-zero precisely when $\phi_2 = \overline{\phi_1}$, and so this simplifies to

$$\mathrm{Corr}(\mu, \nu)[f] = \sum_{\phi \text{ on } \mathbb{G}} \int_{[\mathbb{T}(\mathbb{A})^{\Delta}]} ((Rf'')\overline{\phi} \otimes \phi)(x)dx.$$

Explicitly, the function $((Rf'')\overline{\phi} \otimes \phi)$ is given by

$$\begin{aligned} ((Rf'')\overline{\phi} \otimes \phi)(x) &= \int_{\mathbb{G}(\mathbb{A}) \times \mathbb{G}(\mathbb{A})} f''(s_1, s_2)\overline{\phi(xs_1)}\phi(xs_2)d(s_1, s_2) \\ &= \int f(s_1, s_2)\overline{\phi(xg_1s_1h_1^{-1})}\phi(xg_2s_2h_2^{-1})d(s_1, s_2) \\ &= \int 1_B(s_1)\overline{\phi(xg_1s_1h_1^{-1})}ds_1 \int 1_B(s_2)\phi(xg_2s_2h_2^{-1})ds_2 \end{aligned}$$

which is a function $x \in \mathbb{T}(\mathbb{A})$. It is a product of two forms which are then integrated over the torus in the final summation.

Alternatively, we could consider the double spectral expansion in this case. There are two interesting periods. The non-vanishing of the period $\mathcal{P}_{\mathbb{L}}(\varphi)$ simply tells us that $\varphi = \varphi_1 \otimes \overline{\varphi_1}$. The toric period is then of the form

$$\int_{[\mathbb{T}(\mathbb{A})]} \phi_1(x)\phi_2(x)dx.$$

This approach to joint equidistribution would require a bound on

$$\sum_{\phi_1, \phi_2} \int_{[\mathbb{T}_i(\mathbb{A})]} \phi_1(x)\phi_2(x)dx \sum_{\varphi} a_{\varphi}(R(1_{gB^{(n)}h^{-1}})\overline{\phi_1 \otimes \phi_2})$$

as $n \to \infty$, where ϕ_1, ϕ_2, φ run over the eigenforms of $\mathbb{G}$.

Chapter 7

Analytic Input for Mixing

For the Mixing Conjecture, once we have reduced to a sum of values of a multiplicative function in a number field, there is a reasonable amount of analytic input required to bound these sums. We deal with the required analysis in this chapter. First, we prove a Van der Corput bound on the number of integral points in an ellipsoid for number fields. Then we develop sieve results analogous to those of Chapter 9 of [Kha17], however over a general number field we require an adelic version of sieve theory. Since sieve theory is inherently compatible with an adelic formulation, this requires little extra work. We were, however, unable to find an adelic formulation of sieves in the literature, so this may be of some independent interest.

7.1 Van der Corput for Number Fields

We require a good bound on the number of points in a Λ-ideal $\mathfrak{r}$ with bounded valuations at the infinite places, of a similar quality to the Van der Corput bound for binary quadratic forms over $\mathbb{Q}$. Such a bound is almost certainly contained in the vast literature on lattices and quadratic forms, however since we were unable to find a reference, we prove the result here, using the same method of Van der Corput (although we follow the review of this method in [Gui78]). To get well controlled error terms, we actually deal not with the set of lattice points such that each Archimedean norm is bounded by a fixed constant, but rather the set of lattice points such that the sum of the Archimedean norms is bounded. This shifts the problem from a product of ellipses to an ellipsoid, which is a more regular domain over which to apply the Fourier theoretic methods of Van der Corput. Expanding the domain over which we perform our summation in Proposition 6.1.22 to an ellipsoid rather than a sum of ellipses does no harm.

Let $\mathscr{E}_R = \{(x_\nu)_\nu \in K_\infty : \sum_\nu |x_\nu|_\nu^2 \leq R^2\}$, for $R > 0$, where we use the usual norm on $K_\nu \cong \mathbb{C}$ rather than its square. We wish to get a bound for

$$S(a, \mathcal{I}, \mathfrak{r}, R) := |(a + \mathcal{I}\mathfrak{r}) \cap \mathscr{E}_R| = \sum_{x \in (a+\mathcal{I}\mathfrak{r})} \mathbf{1}_{\mathscr{E}_R}(x),$$

where $\mathfrak{r}$ is a Λ-ideal (Λ an $\mathcal{O}_F$-order in K), $a \in \mathfrak{r}$ and $\mathcal{I} \subset \mathcal{O}_F$ an ideal. The bound that we require is the following: for some $\theta, \eta > 0$, there exists $C > 0$ such that when $\mathrm{covol}(\mathcal{I}\mathfrak{r}) \leq \mathrm{vol}(\mathscr{E})$,

$$\left| S(a, \mathcal{I}, \mathfrak{r}, R) - \frac{\mathrm{vol}(\mathscr{E})}{\mathrm{covol}(\mathcal{I}\mathfrak{r})} \right| \leq C \left(\mathrm{Nm}\mathcal{I}^2\right)^{-\theta} \left(\frac{\mathrm{vol}(\mathscr{E})}{\mathrm{covol}(\mathfrak{r})}\right)^{1-\eta} \tag{7.1}$$

To achieve this, we use Poisson summation for number fields, however we first need to alter the right hand side to be a sum over K of an adelic Schwartz function. This requires some smoothing. Let

$$\rho_0 : \mathbb{R}_{\geq 0} \to R_{\geq 0}$$

be a smooth function such that $\mathrm{supp}(\rho_0) \subset [0, 1]$, all one sided derivatives at $r = 0^+$ are 0, and $\int_0^1 r\rho_0(r)dr = \left(\frac{2\pi^d}{(d-1)!}\right)^{-1}$. Then the radially symmetric function $\rho : \mathbb{R}^{2d} \to \mathbb{R}_{\geq 0}$ defined by $\rho(\underline{x}) = \rho_0(|\underline{x}|)$ is smooth, non-negative, supported on the unit ball, and has total integral 1. Scaling this, we define $\rho_\epsilon(v) = \frac{1}{\epsilon^{2d}}\rho\left(\frac{v}{\epsilon}\right)$. Then, we form the convolution on $K_\infty \cong (\mathbb{R}^2)^{[F:\mathbb{Q}]}$,

$$\mathbf{1}_{R,\epsilon} := \rho_\epsilon * \mathbf{1}_{\mathscr{E}_R},$$

as well as the approximate count

$$S_\epsilon(a, \mathcal{I}, \mathfrak{r}, R) = \sum_{x \in (a+\mathcal{I}\mathfrak{r})} \mathbf{1}_{R,\epsilon}(x).$$

For points inside $\mathscr{E}_{R-\epsilon}$, the function $\mathbf{1}_{R,\epsilon}$ returns identically 1, and for points outside $\mathscr{E}_{R+\epsilon}$ the function returns 0. Therefore, we see that

$$S_\epsilon(a, \mathcal{I}, \mathfrak{r}, R - \epsilon) \leq S(a, \mathcal{I}, \mathfrak{r}, R) \leq S(a, \mathcal{I}, \mathfrak{r}, R + \epsilon).$$

We can write

$$\sum_{x \in (a+\mathcal{I}\mathfrak{r})} \mathbf{1}_{R,\epsilon} = \sum_{x \in K} \Phi(x)$$

for a Bruhat-Schwartz function defined by $\Phi_\infty(x_\infty) = \mathbf{1}_{R,\epsilon}(x_\infty)$, which is a Archimedean Schwartz function (actually a uniform limit of such things, however the Poisson Summation formula also holds for these) due to the smoothing of ρ_{ϵ_ν}, and $\Phi_f(x_f) = \mathbf{1}_{a+(\mathcal{I}\mathfrak{r})\otimes\widehat{\mathbb{Z}}}(x_f)$. The adelic Poisson Summation formula now tells us that

$$\sum_{x\in K} \Phi(x) = \sum_{x\in K} \widehat{\Phi}(x).$$

A simple computation of the Fourier transform of the Bruhat-Schwartz function Φ shows that this identity is

$$S_\epsilon(a,\mathcal{I},\mathfrak{r},R) = \frac{2^{[F:\mathbb{Q}]}}{|D_\Lambda|^{1/2}\mathrm{Nm}\mathcal{I}^2\mathrm{Nm}\mathfrak{r}} \sum_{x\in\mathcal{I}^{-1}\widehat{\mathfrak{r}}} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(xa)}\widehat{\mathbf{1}_{\mathscr{E}_R}}(x)\widehat{\rho_\epsilon}(x).$$

The product in the infinite places arises since the Fourier transform of a convolution is simply a product of the Fourier transforms. The $x = 0 \in \widehat{\mathfrak{r}}$ term gives the leading order growth

$$\frac{2^{[F:\mathbb{Q}]}\widehat{\mathbf{1}_{\mathscr{E}_R}}(0)\widehat{\rho_\epsilon}(0)}{|D_\Lambda|^{1/2}\mathrm{Nm}\mathcal{I}^2\mathrm{Nm}\mathfrak{r}} = \frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathcal{I}\mathfrak{r})}.$$

Thus it remains to bound the error term

$$E_\epsilon(a,\mathcal{I},\mathfrak{r},R) := \sum_{x\in\mathcal{I}^{-1}\widehat{\mathfrak{r}}\setminus 0} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(xa)}\widehat{\mathbf{1}_{\mathscr{E}_R}}(x)\widehat{\rho_\epsilon}(x).$$

First, since ρ_ϵ is a smooth compactly supported function, its Fourier transform is rapidly decaying faster than any non-zero rational function. Also,

$$\widehat{\rho_\epsilon}(x) = \widehat{\rho}(\epsilon x).$$

Thus, for every $N \in \mathbb{N}$ we can find an $A > 0$ depending only on N such that

$$\widehat{\rho_\epsilon}(x) \le A(1+\epsilon^2\sum_\nu |x_\nu|_\nu^2)^{-N}, \forall \epsilon\in\mathbb{R}_{>0}, x\in K_\infty. \tag{7.2}$$

For the other term, we can use the standard computation of the Fourier transform of the indicator function of the unit ball to get

$$\widehat{\mathbf{1}_{\mathscr{E}_R}}(x) = R^d \frac{J_d(R\sqrt{\sum_\nu |x_\nu|_\nu^2})}{(\sum_\nu |x_\nu|^2)^{d/2}},$$

where J_d is the order $d = [F : \mathbb{Q}]$ Bessel function. There exists a constant $B > 0$ such that for all $z > 0$, $J_d(z) \le Bz^{-1/2}$. Therefore, we deduce that there is a fixed constant $C > 0$ depending only on N, such that

$$\widehat{\mathbf{1}_{\mathscr{E}_R}}(x)\widehat{\rho_\epsilon}(x) \le CR^{d-1/2}\frac{(1+\epsilon^2\sum_\nu |x_\nu|_\nu^2)^{-N}}{(\sum_\nu |x_\nu|^2)^{d/2+1/4}}.$$

Suppose that we choose a symmetric fundamental domain $\mathcal{F}$ for the lattice $\widehat{\mathfrak{r}}$, and let $r_1, r_2 > 0$ be optimal such that $B(r_1) \subset \mathcal{F} \subset B(r_2)$. Let

$$f_\epsilon(x) = \frac{(1 + \sum_\nu \epsilon^2 |x_\nu|^2)^{-N}}{(\sum_\nu |x_\nu|^2)^{d/2+1/4}}.$$

Then, we may approximate

$$f_\epsilon(x) \leq \frac{1}{\mathrm{vol}\mathcal{F}} \int_{y \in x+\mathcal{F}} f_\epsilon(y) dy \sup_{y \in x+\mathcal{F}} \frac{f_\epsilon(x)}{f_\epsilon(y)}.$$

Since the supremum on the right is reached when $||x||_2 > 2r_1$ is minimal, we see that it is uniformly bounded above by $(r_2/r_1)^{d+1/2}$ with the constant depending only on N. We now combine all of the integrals to get an upper bound

$$\left| \sum_{x \in (b+\widehat{\mathfrak{r}}) \setminus 0} \widehat{\mathbf{1}_{\mathscr{E}_R}}(x) \widehat{\rho}_\epsilon(x) \right| \leq A \frac{R^{d-1/2} r_2^{d+1/2}}{\mathrm{vol}(\mathcal{F}) r_1^{d+1/2}} \int_{\mathbb{R}^{2d}} f_\epsilon(y) dy$$

By a change of variables, we finally arrive at

$$E_\epsilon(a, \mathcal{I}, \mathfrak{r}, R) = A \frac{R^{d-1/2} \epsilon^{1/2-d} r_2^{d+1/2}}{\mathrm{vol}(\mathcal{F}) r_1^{d+1/2}} \sum_{x \in \mathcal{I}^{-1}\widehat{\mathfrak{r}}/\widehat{\mathfrak{r}}} w_x e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(xa)} \tag{7.3}$$

where $w_x \in (-1, 1)$. Let $G_{w,a,\mathcal{I},\mathfrak{r}}$ be the exponential sum given here. Notice that the weights here are not arbitrary, they are the weights of equation (7.2) scaled by a common factor.

To get the optimal bound, we balance the errors in physical space with those in reciprocal (Fourier) space. This leads to us setting

$$\frac{R^{d-1/2} \epsilon^{1/2-d} r_2^{d+1/2}}{\mathrm{vol}(\mathcal{F}) r_1^{d+1/2}} G_{w,a,\mathcal{I},\mathfrak{r}} = R^{2d-1} \epsilon,$$

which gives

$$\epsilon = R^{-\frac{2d-1}{2d+1}} \left(\frac{r_2}{r_1} \right) \mathrm{vol}(\mathcal{F})^{-\frac{2}{2d+1}} G_{w,a,\mathcal{I},\mathfrak{r}}^{\frac{2}{2d+1}}.$$

The two errors now match, and we get a single error term. Noting that $\mathrm{vol}(\mathcal{F}) = |D_\Lambda|^{-1/2} \mathrm{Nm}\mathfrak{r}^{-1}$, we reach

$$S(a, \mathcal{I}, \mathfrak{r}, R) = \frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathcal{I}\mathfrak{r})} + O\left(\left(\frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathcal{I}\mathfrak{r})} \right)^{\frac{2d-1}{2d+1}} \left(\frac{G_{w,a,\mathcal{I},\mathfrak{r}}^2}{\mathrm{Nm}\mathcal{I}^4} \right)^{\frac{1}{2d+1}} \left(\frac{r_2}{r_1} \right) \right)$$

Now, in order to deduce the bound of equation (7.1) that we require in the next section, we must prove

$$\left(\frac{r_2}{r_1} \right) |G_{w,a,\mathcal{I},\mathfrak{r}}|^{\frac{2}{2d+1}} \ll (\mathrm{Nm}\mathcal{I}^2)^{\frac{2}{2d+1}-\theta} \left(\frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathfrak{r})} \right)^{\frac{2}{2d+1}-\eta}$$

for some $\theta, \eta > 0$. We expect the following:

Assumption 7.1.1. *A bound of the form*

$$|G_{w,a,\mathcal{I},\mathfrak{r}}| \leq \left(\mathrm{Nm}\mathcal{I}^2\right)^{1-\theta}$$

when $\mathrm{Nm}\mathcal{I}^2 \leq \frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathfrak{r})}$.

However we have been unable to find this in the literature in this setting and so simply list it as an assumption to the main theorem. Given this, to get a non-trivial point count bound we require a result of the form

$$\frac{r_2}{r_1} \leq \left(\frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathfrak{r})}\right)^{\frac{2}{2d+1}-\eta_0}.$$

If we translate this into the setting for which we will be applying the result, i.e. to the counts given in Proposition 6.1.22, i.e. where $\mathfrak{r} = \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}$ and $R^2 \asymp \mathrm{Nm}(\mathfrak{s})^{-1/d}$, then

$$\frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathfrak{r})} = A|D_\Lambda|^{1/2}.$$

Therefore the required result is

$$\frac{r_2}{r_1} \ll |D_\Lambda|^{\theta_l}$$

for some $\theta_l > 0$. Recall from the method above that r_1, r_2 are chosen optimally such that $B(r_1) \subset \mathcal{F}_{\widehat{\mathfrak{r}}} \subset B(r_2)$ where $\mathcal{F}_{\widehat{\mathfrak{r}}}$ is a fundamental domain for $\widehat{\mathfrak{r}} \subset K_\infty = \mathbb{C}^d$ which we may choose.

As this is a statement on the shape of fundamental domains of $\mathcal{O}_F$-lattices in K, it is natural to use reduction theory for $\mathrm{SL}_2(F)/\mathrm{SL}_2(\mathcal{O}_F)$. We summarise this theory in the proposition below (following [Str21]). Let $\mathfrak{c}_1, ..., \mathfrak{c}_h \subset \mathcal{O}_F$ be the integral representatives of minimal norm in $\mathcal{O}_F$ for each class in Cl_F. Since ideals in Dedekind domains can always be generated by two elements, we write $\mathfrak{c}_i = (\rho_i, \sigma_i)$ - and fix elements $\eta_i, \xi_i \in \mathfrak{c}_i^{-1}$ such that $\rho_i\eta_i - \sigma_i\xi_i = 1$. In addition, we choose a basis $\epsilon_1, ..., \epsilon_{d-1}$ of fundamental units for the group $\mathcal{O}_F^\times/\{\pm 1\}$. Finally, for each ideal $\mathfrak{c}_i$, choose an integral basis $\beta_1^i, ..., \beta_d^i$ for $\mathfrak{c}_i^{-2}$. Recall also that we have identified $K_\infty \cong \mathbb{C}^d$ and inside this space we have the upper-half space $\mathbb{H}^d = \{(z_\nu)_\nu : \mathrm{Im}(z_\nu) >, \forall \nu\}$.

Proposition 7.1.2. *Let* $\mathfrak{s} \subset K$ *be a* Λ*-ideal (recall* Λ *is an* $\mathcal{O}_F$*-order). Then there is a free* $\mathcal{O}_F$*-submodule* $\mathfrak{s}' \subset \mathfrak{s}$ *with index at most* $C := \max_i N\mathfrak{c}_i$*, with an* $\mathcal{O}_F$*-basis* $\{x, zx\} \subset \mathfrak{s}'$ *satisfying the following for some ideal class* $\mathfrak{c}_i$*:*

- $z \in K \cap \mathbb{H}^d$*;*

- *The function*

$$\begin{aligned} \{(\mathcal{I}, v) : v \in \mathcal{I}\mathfrak{s}'\} &\longrightarrow \mathbb{Q}_{>0} \\ (\mathcal{I}, v) &\longmapsto \mathrm{Nm}(v)\mathrm{Nm}(\mathcal{I})^{-2} \end{aligned}$$

  where $\mathcal{I}$ is an integral ideal of $\mathcal{O}_F$, is minimised at $(\mathfrak{c}_i, \rho_i x - \sigma_i z x)$.

- $(\mathrm{Re}(z_\nu))_\nu \in \sum_{j=1}^{d}[-\frac{1}{2}, \frac{1}{2})\beta_j^i$.

- $(\log(\mathrm{Im}(z_\nu)))_\nu \in \frac{1}{d}\log(\prod_\nu \mathrm{Im}(z_\nu)) + \sum_{j=1}^{d-1}[-1, 1)(\log(|\epsilon_j|_\nu))_\nu$.

Proof. The first statement follows from the fact that by the classification theorem for modules over a Dedekind domain,

$$\mathfrak{s} \cong \mathcal{O}_F \oplus \mathfrak{c}_i$$

for a unique $\mathfrak{c}_i$. This contains the submodule $\mathfrak{c}_j \oplus \mathfrak{c}_i$ where $\mathfrak{c}_j$ is the representative of the class $[\mathfrak{c}_i]^{-1}$. Now by Steinitz's Theorem, $\mathfrak{c}_j \oplus \mathfrak{c}_i$ is free of rank 2.

The conditions on $\mathfrak{s}'$ are simply a summary of [Str21]. □

Part of the construction of this fundamental domain is the notion of the distance to a cusp. Let $(\rho, \sigma) = \mathfrak{a}$ be a cusp (with $\rho, \sigma \in \mathcal{O}_F$). Then the distance $\Delta(z, (\rho, \sigma))$ is defined to be

$$\Delta(z, (\rho, \sigma)) := N(\mathfrak{a})^{-1} N(\mathrm{Im}(z))^{-1/2} N(\rho - \sigma z)^{1/2}.$$

The second condition in the Proposition is equivalent to the fact that $(\rho_i, \sigma_i) = \mathfrak{c}_i$ is the nearest cusp.

Definition 7.1.3. *For any compact subset, $\mathcal{C}$, in $\mathbb{R}^{2d}$, let $r_1(\mathcal{C}), r_2(\mathcal{C})$ be the smallest (resp. largest) radius of a circle containing (resp. contained in) $\mathcal{C}$.*

Definition 7.1.4. *Let $\mathcal{E}_0$ be the sphere in $\mathbb{R}^{2d}$ with centre at the origin and radius $1/2$. Given a basis, $\mathscr{B}$, of $\mathbb{R}^{2d}$, define the ellipsoid, $\mathcal{E}_{\mathscr{B}}$, generated by $\mathscr{B}$ to be the image of $\mathcal{E}_0$ under the unique transformation taking the standard basis to $\mathscr{B}$.*

Lemma 7.1.5. *Suppose that $z \in \Psi$ where $\Psi \subset \mathbb{H}^d$ satisfies*

1. *for each $i = 1, ..., d$, there exists an $\epsilon_i > 0$ such that $\forall w \in \Psi$, $\mathrm{Im}(w_i) > \epsilon_i$ for all i, and there is a constant B such that $\mathrm{Im}(w_i)/\mathrm{Im}(w_j) < B$ for all i, j and all $w \in \Psi$.*

2. *the projection $\mathrm{Re} : \Psi \to \mathbb{R}^d$ has bounded image.*

Then, there is a constant A_Ψ *such that for every ellipsoid,* $\mathcal{E}_z$*, generated by the basis* $\mathscr{B}_{\beta,z} := \{\beta_1, ..., \beta_d, \beta_1 z, ..., \beta_d z\}$ *satisfies*

$$\frac{r_2(\mathcal{E}_{\mathscr{B}_{\beta,z}})}{r_1(\mathcal{E}_{\mathscr{B}_{\beta,z}})} \leq A_\Psi \left(\prod_i \mathrm{Im}(z_i)\right)^{1/d}.$$

Here, $\{\beta_1, ..., \beta_d\} = \{\beta_1^1, ..., \beta_d^1\}$ *is the chosen fixed basis of* $\mathcal{O}_F$*.*

Proof. Let $\underline{\beta}$ be the matrix given by the embeddings of β_i into $\mathbb{R}^d$, that is, if $\beta_i = (\beta_{i,1}, ..., \beta_{i,d}) \in \mathbb{R}^d$,

$$\underline{\beta} = \begin{pmatrix} \beta_{1,1} & \dots & \beta_{d,1} \\ \vdots & \ddots & \vdots \\ \beta_{1,d} & \dots & \beta_{d,d} \end{pmatrix}.$$

Also, let $\underline{x} = \mathrm{diag}(\mathrm{Re}(z_1), ..., \mathrm{Re}(z_d))$ and $\underline{y} = \mathrm{diag}(\mathrm{Im}(z_1), ..., \mathrm{Im}(z_d))$. Then the matrix sending the standard basis of $\mathbb{C}^d$ to the basis of the Lemma is given by

$$\begin{pmatrix} \underline{\beta} & \underline{x\beta} \\ 0 & \underline{y\beta} \end{pmatrix} = \begin{pmatrix} \underline{\beta} & \\ & \underline{\beta} \end{pmatrix} \begin{pmatrix} 1_d & \underline{x} \\ & \underline{y} \end{pmatrix}.$$

Therefore, the ellipsoid $\mathcal{E}_z$ is the image under the matrix $\underline{Z} = \begin{pmatrix} 1_d & \underline{x} \\ & \underline{y} \end{pmatrix}$ of the fixed ellipsoid $\mathcal{E}_1$, which is the ellipsoid generated by the $\mathbb{R}$-basis $\{\beta_1, ..., \beta_d, \beta_1 i, ..., \beta_d i\}$ of $\mathbb{C}^d$. The quantities $r_1(\mathcal{E}_1), r_2(\mathcal{E}_1)$ depend only on the chosen basis for F. Let $\mathscr{B}_z$ be the basis $\{e_1, ..., e_d, z_1, ..., z_d\}$ of $\mathbb{C}^d$. Then it is clear that

$$r_1(\mathcal{E}_1) r_1(\mathcal{E}_{\mathscr{B}_z}) \leq r_1(\mathcal{E}_{\mathscr{B}_{\beta,z}}) \leq r_2(\mathcal{E}_{\mathscr{B}_{\beta,z}}) \leq r_2(\mathcal{E}_1) r_2(\mathcal{E}_{\mathscr{B}_z}).$$

Therefore,

$$\frac{r_2(\mathcal{E}_{\mathscr{B}_{\beta,z}})}{r_1(\mathcal{E}_{\mathscr{B}_{\beta,z}})} \ll_{F,\underline{\beta}} \frac{r_2(\mathcal{E}_{\mathscr{B}_z})}{r_1(\mathcal{E}_{\mathscr{B}_z})}. \tag{7.4}$$

To analyse $\mathcal{E}_{\mathscr{B}_z}$ it makes sense to change the order of the basis such that the matrix moving the standard basis to this one is block diagonal with 2x2 block entries,

$$M = \begin{pmatrix} 1 & \mathrm{Re}(z_1) & & & \\ & \mathrm{Im}(z_1) & & & \\ & & \ddots & & \\ & & & 1 & \mathrm{Re}(z_d) \\ & & & & \mathrm{Im}(z_d) \end{pmatrix}.$$

The ellipsoid $\mathcal{E}_{\mathscr{B}_z}$ is equal to

$$\left\{ v^T (M^{-1})^T M^{-1} v = 1/4 \right\},$$

and therefore we need to determine the ratio of the (square roots of the) largest and smallest eigenvalue of $(M^{-1})^T M^{-1}$, or equivalently of MM^T. This is block diagonal again with blocks

$$\begin{pmatrix} 1+\mathrm{Re}(z_i)^2 & \mathrm{Re}(z_i)\mathrm{Im}(z_i) \\ \mathrm{Re}(z_i)\mathrm{Im}(z_i) & \mathrm{Im}(z_i)^2 \end{pmatrix}$$

Let $\lambda_{i,big} > \lambda_{i,small} > 0$ be the eigenvalues of this matrix. Then

$$\frac{\lambda_{big}}{\mathrm{Im}(z_i)^2} = \frac{1}{2}\left(1+\mathrm{Im}(z_i)^{-2}(1+\mathrm{Re}(z_i)^2)\right)+\sqrt{\left(1+\mathrm{Im}(z_i)^{-2}(1+\mathrm{Re}(z_i)^2)\right)^2 - 4\mathrm{Im}(z_i)^{-2}}.$$

Since $0 \le \mathrm{Im}(z_i)^{-1} < \epsilon_i^{-1}$ and $\mathrm{Re}(z_i)$ ranges over a bounded subset of $\mathbb{R}$, this quantity is bounded above independently of z. Therefore,

$$\sqrt{\frac{\lambda_{big}}{\lambda_{small}}} = \sqrt{\frac{\lambda_{big}^2}{\mathrm{Im}(z_i)^2}} \le B_\Psi \mathrm{Im}(z_i).$$

In fact

$$\frac{r_2(\mathcal{E}_{\mathscr{B}_z})}{r_1(\mathcal{E}_{\mathscr{B}_z})} = \sqrt{\frac{\max_i \lambda_{i,big}}{\min_i \lambda_{i,small}}}.$$

However, using the first condition of the Lemma, we see that for any i, j,

$$\begin{aligned} \frac{\lambda_{i,big}}{\lambda_{j,small}} &= \frac{\lambda_{i,big}\lambda_{j,big}}{\lambda_{j,small}\lambda_{j,big}} \\ &\le B^2 \frac{\lambda_{i,big}\lambda_{j,big}}{\lambda_{i,big}\lambda_{i,small}} \\ &= B^2 \frac{\lambda_{j,big}}{\lambda_{i,small}}. \end{aligned}$$

If we choose i, j such that $\lambda_{i,big}$ is maximal and $\lambda_{j,small}$ is minimal, then

$$\begin{aligned} \frac{r_2(\mathcal{E}_{\mathscr{B}_z})}{r_1(\mathcal{E}_{\mathscr{B}_z})} = \sqrt{\frac{\lambda_{i,big}}{\lambda_{j,small}}} \le B\sqrt{\frac{\lambda_{j,big}}{\lambda_{i,small}}} \le B\sqrt{\frac{\lambda_{i,big}}{\lambda_{i,small}}} \\ \le B_\Psi B \mathrm{Im}(z_i) \le B_\Psi B^{2-\frac{1}{d}} \left(\prod_i \mathrm{Im}(z_i)\right)^{1/d}. \end{aligned}$$

Combining this with (7.4) we complete the proof. □

Proposition 7.1.6. *The ratio r_2/r_1 for the fundamental domain $\mathcal{F}$ of the lattice $\mathfrak{s}$ is bounded above by*

$$\frac{r_2}{r_1} \ll_F \left(\frac{\mathrm{covol}(\mathfrak{s})}{\min_{v\in\mathfrak{s}} N(v)}\right)^{1/d}.$$

Proof. We can choose the fundamental domain $\mathcal{F}$ to be the image under a unique element of $\mathrm{GL}_{2d}(\mathbb{R})$ of the (closed) fundamental domain

$$\mathcal{F}_0 = \left\{ (x_i)_i : |x_i| \leq \frac{1}{2} \right\}$$

for the lattice $\mathbb{Z}^{2d} \subset \mathbb{R}^{2d} = \mathbb{C}^d = K_\infty$. Suppose furthermore that the basis corresponding to this transformation is of the form $\mathscr{B}_{\beta,z}$ for some $z \in \mathbb{H}^d$ as defined in the previous Lemma. Consider the ellipsoid, $\mathcal{E}$, contained in $\mathcal{F}$, which is the image of the sphere of radius $1/2$ under this same map. It is easy to see that

$$\mathcal{E} \subset \mathcal{F} \subset \sqrt{2d}\mathcal{E}$$

Therefore,

$$(2d)^{-1/2}\frac{r_1(\mathcal{E})}{r_2(\mathcal{E})} \leq \frac{r_1(\mathcal{F})}{r_2(\mathcal{F})} \leq (2d)^{1/2}\frac{r_1(\mathcal{E})}{r_2(\mathcal{E})}.$$

By Corollary 9 of [Str21], we know that the fundamental domain described in Proposition 7.1.2 is the union of a compact part and the cuspidal parts associated to $\mathfrak{c}_i = (\rho_i, \sigma_i)$ for each element of the class group of F. There exists a fixed constant $D > 0$ depending only on F such that if $\mathfrak{c}_i$ is the closest cusp to z then $\Delta(z, \mathfrak{c}_i) < D$.

Then, consider the lattice $\mathfrak{s}'' = \langle \eta_i xz - \xi_i x, -\sigma_i xz + \rho_i x \rangle$. This satisfies

$$\mathfrak{c}_i\mathfrak{s}' \subset \mathfrak{s}'' \subset \mathfrak{c}_i^{-1}\mathfrak{s}'$$

Furthermore, we can associate to $\mathfrak{s}''$ the element $w = \frac{\eta_i z - \xi_i}{-\sigma_i z + \rho_i}$. By the definition of $\Delta(z, \mathfrak{c}_i)$, we see that

$$\prod_i \mathrm{Im}(w_i) = \frac{N(\mathrm{Im}(z))}{N(-\sigma_i + \rho_i)} > (N(\mathfrak{c}_i)D)^{-2}.$$

By reducing the basis $1, w$ with respect to the stabiliser in $\mathrm{SL}_2(\mathcal{O}_F)$ of the cusp at ∞, we can conclude that every lattice $\mathfrak{s}$ has a commensurable lattice $\mathfrak{s}''$ with

$$[\mathfrak{s} : \mathfrak{s}'' \cap \mathfrak{s}], [\mathfrak{s}'' : \mathfrak{s}'' \cap \mathfrak{s}] \leq C^2,$$

such that $\mathfrak{s}''$ is a free $\mathcal{O}_F$-lattice of rank 2 generated by $\{x', z'x'\}$ with $z' \in \Psi$ where Ψ is as in Lemma 7.1.5 with the constants $(\epsilon_i)_i, B$ depending only on F.

Moving to a commensurable lattice in this way loses at most a factor of C^2 to the best possible ratio r_2/r_1 for the lattice.

Thus, by the Lemma 7.1.5, we get

$$\frac{r_1(\mathcal{F})}{r_2(\mathcal{F})} \ll_F N(\mathrm{Im}(z'))^{1/d} = \mathrm{covol}(\mathfrak{s})^{1/d} N(x')^{-1/d}.$$

From the construction above, we see that $x' = (-\sigma_i z + \rho_i)x$ and by the definition of $\mathfrak{c}_i = (\rho_i, \sigma_i)$ being the nearest cusp, we see that

$$\min_{v \in \mathfrak{s}'} N(v) \le N(x') \le N(\mathfrak{c}_i)^2 \min_{v \in \mathfrak{s}'} N(v)$$

Therefore $N(x') \asymp_F \min_{v \in \mathfrak{s}} N(v)$. The result follows. □

Corollary 7.1.7. *In the setting of Proposition 6.1.22, let* $\mathfrak{R}_\mathfrak{c} = \min_{\substack{\mathfrak{a} \subset \Lambda \\ [\mathfrak{a}] = [\mathfrak{c}^{-1}\mathfrak{s}]}} N\mathfrak{a}$. *Suppose there exists* $\eta > 0$ *such that*

$$\mathfrak{R}_\mathfrak{c} \ge |D_\Lambda|^{\frac{1}{2(2d+1)} + \eta}$$

then

$$\left| S(a, \mathcal{I}, \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}, R) - \frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathcal{I}\mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda})} \right| \ll (\mathrm{N}\mathcal{I}^2)^{-\theta} \left(\frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda})} \right)^{1-\eta},$$

where $\theta > 0$ *comes from the Van der Corput bound on exponential sums from Assumption 7.1.1.*

Proof. In this setting, $\mathfrak{r} = \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda}$, $R^2 \asymp \mathrm{N}(\mathfrak{s})^{-1/d}$ and

$$\frac{\mathrm{vol}(\mathscr{E}_R)}{\mathrm{covol}(\mathfrak{r})} \asymp |D_\Lambda|^{1/2}.$$

Applying the previous Proposition to $\widehat{\mathfrak{r}} = \mathfrak{c}^{-1}\mathfrak{s}$, we get

$$\frac{r_2(\mathcal{F}_{\widehat{\mathfrak{r}}})}{r_1(\mathcal{F}_{\widehat{\mathfrak{r}}})} \ll_F \left(\frac{|D_\Lambda|^{1/2} N(\mathfrak{c})^{-2} N(\mathfrak{s})}{\min_{v \in \mathfrak{c}^{-1}\mathfrak{s}} N(v)} \right)^{1/d} = \left(\frac{|D_\Lambda|^{1/2}}{\mathfrak{R}_\mathfrak{c}} \right)^{1/d}$$

Therefore, the error term in the Van der Corput method is bounded uniformly by

$$\left(|D_\Lambda|^{1/2}\right)^{\frac{2d-1}{2d+1}} \frac{G^{\frac{2}{2d+1}}}{N\mathcal{I}^2} |D_\Lambda|^{\frac{1}{2d}} \mathfrak{R}_\mathfrak{c}^{-1/d} \le (N\mathcal{I}^2)^{-\theta} |D_\Lambda|^{\frac{1}{2}(1-\eta)},$$

where the exponent $\theta > 0$ comes from Van der Corput's bound on exponential sums. □

Suppose that $\mathfrak{R}_F = \max_{[\mathfrak{c}] \in \mathrm{Cl}_F} \min_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ [\mathfrak{a}] = [\mathfrak{c}]}} N\mathfrak{a}$, then clearly

$$\mathfrak{R}_{[\mathcal{O}_F]} \mathfrak{R}_F^{-2} \le \mathfrak{R}_\mathfrak{c} \le \mathfrak{R}_{[\mathcal{O}_F]} \mathfrak{R}_F^2$$

Therefore, for F fixed, the assumption of Corollary 7.1.7 on $\mathfrak{R}_\mathfrak{c}$ is equivalent to the same assumption for any other $\mathfrak{c}' \in \mathrm{Cl}_F$. Therefore we may simply make the assumption on $\mathfrak{R} := \mathfrak{R}_\mathfrak{c}$.

7.2 Sieving

Our application of the large sieve is largely the same as in [Kha17]. One key difference is that we are no longer analysing a multiplicative function on $\mathcal{O}_F$, but rather one on ideals of $\mathcal{O}_F$. It is natural to replace with a multiplicative function on the adeles. We are unaware of previous applications of sieve theory to adelic functions, however the sieving process is very nicely adapted to this setting due to the conditions placed at each prime separately. Once the actual sieving procedure is underway, it is almost identical to Section 9 of [Kha17].

The method is to use the large sieve of Kowalski [Kow08], and split the sum on the left into different regions depending on the prime factorisation of $Q(x, y)$. In the language of Kowalski, we give the set up of the prime sieve below.

Let $\mathfrak{r} = \cap_{\nu\nmid\infty} r_\nu \Lambda_\nu \subset K$ be a proper fractional Λ-ideal, $\mathfrak{b} \subset \mathcal{O}_F$ an integral $\mathcal{O}_F$-ideal, and $x_0 \in \mathfrak{r}$. Consider a map

$$Q : x_0 + \mathfrak{b}\mathfrak{r} \to \prod_{\nu\nmid\infty} \mathcal{O}_{F_\nu}$$

such that each component of the map can be written as a quadratic polynomial after choosing an $\mathcal{O}_{F_\nu}$-basis of $\mathfrak{b}\mathfrak{r}_\nu$. For our application, we will use the map of the form

$$Q(x)_\nu = \frac{\mathrm{Nm}(x) - y}{\mathrm{Nm}(\mathfrak{r})_\nu}$$

for some fixed $y \in \mathrm{Nm}(\mathfrak{r})$. It is clear that we can reduce the map Q modulo any ideal of $\mathcal{O}_F$, to get a map

$$Q_\mathfrak{a} : x_0 + \mathfrak{b}\mathfrak{r}/\mathfrak{a}\mathfrak{b}\mathfrak{r} \to \mathcal{O}_F/\mathfrak{a}.$$

Fundamentally, the sieving process depends only of the maps $Q_\mathfrak{a}$ which is why the adelic process is so similar to the rational process. The sieve setting is as follows (note that it depends on the Λ-ideal $\mathfrak{r} \subset K$ and $z > 0$, however as far as possible the sieve bounds derived from this setting will show transparently how they depend on this data).

- The sieve setting is

  $$\Psi = (x_0 + \mathfrak{b}\mathfrak{r}, \Sigma_F, (\rho_\mathfrak{p} : \mathfrak{b}\mathfrak{r} \to (\mathfrak{b}\mathfrak{r}/\mathfrak{b}\mathfrak{r}\mathfrak{p}))_{\mathfrak{p}\in\Sigma_F}).$$

  For convenience, from now on we write $\mathbb{F}_\mathfrak{p}(\mathfrak{r}) := (\mathfrak{r}/\mathfrak{r}\mathfrak{p})$ in parallel with the standard notation $\mathcal{O}_F/\mathfrak{p} =: \mathbb{F}_\mathfrak{p}$, however $\mathbb{F}_\mathfrak{p}(\mathfrak{r})$ doesn't come with a naturally defined field structure.

- The siftable set is
$$\Upsilon = (\mathscr{E} \cap (x_0 + \mathfrak{br}), \mu_{\text{disc}}, \text{incl.})$$
where $\mathscr{E} = \left\{(x_\nu)_{\nu|\infty} : \sum_\nu |x_\nu|^2 \le R^2\right\}$ is the ellipsoid with bounded 2-norm.

- The prime sieve support is
$$\mathcal{L}^* = \{\mathfrak{p} \in \Sigma_F : N\mathfrak{p} \le z\},$$
where $z \in \mathbb{R}_{>0}$ is yet to be specified.

- The sieving sets are
$$\Omega_\mathfrak{p} = \{x_0 + x + \mathfrak{rbp} \in \mathfrak{rb}/\mathfrak{rbp} : Q_\mathfrak{p}(x) = 0\}.$$

- The sifted sets are therefore
$$S(\mathscr{E} \cap (x_0 + \mathfrak{rb}), \Omega; \mathcal{L}^*) = \{x \in \mathscr{E} \cap (x_0 + \mathfrak{br}) : Q_\mathfrak{p}(x) \ne 0, \forall \mathfrak{p} \in \mathcal{L}^*\}.$$

- Finally, we take our sieve support to be
$$\mathcal{L} = \{\text{square free ideals } \mathcal{I} \subset \mathcal{O}_F : N\mathcal{I} \le z\}.$$
From here on in this section, unless otherwise stated, ideal means square-free ideal.

That is, the large sieve will give us estimate for the number of points $x \in \mathscr{E} \cap (x_0 + \mathfrak{br})$ such that $\mathrm{Nm}(\mathfrak{r})^{-1}(\mathrm{Nm}(x) - y)$ is z-rough (all its prime factors have norm least z). This bound will be in terms of the large sieve constant Δ. A priori, the sieve constant depends on $x_0 + \mathfrak{br}$, but not on Q, since the only dependence on Q is via the sieving sets, which do not effect the large sieve constant. First we need some more definitions.

Definition 7.2.1. *For any ideal $\mathfrak{a} \subset \mathcal{O}_F$, define $\rho_Q(\mathfrak{a})$ to be the number of solution to the equation $Q_\mathfrak{a}(x) = 0$ (recall, $Q_\mathfrak{a}$ is a function on $\mathfrak{br}/\mathfrak{abr}$).*

For any prime power $\mathfrak{p}^k$ let $\widetilde{\rho}_Q(\mathfrak{p}^k)$ denote the number of solutions to $Q_{\mathfrak{p}^k}(x) = 0$ which have either 0 or $\mathrm{N}\mathfrak{p}$ lifts to solutions over $\mathfrak{br}/\mathfrak{p}^{k+1}\mathfrak{br}$. Extend this to a multiplicative function on ideals of $\mathcal{O}_F$.

Definition 7.2.2. *Let $A \ge 1, B, \epsilon > 0$. We say that a multiplicative function $f : \prod_{\nu\nmid\infty} \mathcal{O}_{F_\nu} \to \mathbb{R}$ is of class $\mathscr{M}(A, B, \epsilon)$ if it is non-negative and for any $x = (x_\nu)_\nu$*
$$f(x) \le \min\left(A^{\Omega(x)}, B(Nx)^\epsilon\right).$$
Here $\Omega(x) = \sum_\nu v_\nu(x_\nu)$ denotes the number of prime factors of n counted with multiplicity, and $Nx = \prod_\nu N\mathfrak{p}_\nu^{v_\nu(x_\nu)}$ is the norm.

Assumption 7.2.3. *Assume that there exists an $\eta, \theta > 0$ such that for some constant $C > 0$, and any $\mathcal{I} \subset \mathcal{O}_F$ with $\mathrm{Nm}\mathcal{I}^2 \leq \frac{\mathrm{vol}(\mathscr{E})}{\mathrm{covol}(\mathfrak{r})}$ and $y \in \mathfrak{r}$,*

$$\left| |\mathscr{E} \cap (y + \mathcal{I}\mathfrak{r})| - \frac{\mathrm{vol}(\mathscr{E})}{\mathrm{covol}(\mathcal{I}\mathfrak{r})} \right| \leq C \left(\mathrm{Nm}\mathcal{I}^2\right)^{-\theta} \left(\frac{\mathrm{vol}(\mathscr{E})}{\mathrm{covol}(\mathfrak{r})}\right)^{1-\eta}$$

uniformly for all considered $(\mathscr{E}, \mathfrak{r})$. From Section 7.1, we see that θ is the best cancellation of exponential sums available, and η is provided by the reduction theory of $\mathrm{SL}_2(F)$ along with an assumption of a lower bound for $\mathfrak{R}_\mathfrak{c}$. As such, we can assume that $\theta < 1$.

Notation 7.2.4. *From now on, let*

$$A_{\mathscr{E},\mathfrak{r}} := \frac{\mathrm{vol}(\mathscr{E})}{\mathrm{covol}(\mathfrak{r})}.$$

This denotes the expected number of points in $\mathscr{E} \cap \mathfrak{r}$, as per the assumption above.

Applying the equidistribution bound from [Kow08, Corollary 2.13], we get that (using the same notation that $[\mathcal{I}, \mathcal{J}]$ is the square-free ideal with support given by the union of the supports of $\mathcal{I}$ and $\mathcal{J}$), provided

$$1 \leq z \leq \min\left(A_{\mathscr{E},\mathfrak{r}}^{1/4} N\mathfrak{b}^{-2}, \left(A_{\mathscr{E},\mathfrak{r}}^{\eta} N\mathfrak{b}^{2(\theta-1)}\right)^{1/(3d+3)}\right),$$

the sieve constant can be bounded by

$$\begin{aligned}
\Delta - |\mathscr{E} \cap (x_0 + \mathfrak{b}\mathfrak{r})| &\leq \max_{N\mathcal{J} \leq z} \sum_{N\mathcal{I} \leq z} \sum_{y \in \mathfrak{b}\mathfrak{r}/[\mathcal{I},\mathcal{J}]\mathfrak{b}\mathfrak{r}} |r_{[\mathcal{I},\mathcal{J}]}(y)| N\mathcal{I} \\
&\leq 2CN\mathfrak{b}^{-2\theta} \max_{N\mathcal{J} \leq z} \sum_{N\mathcal{I} \leq z} N\mathcal{I} \left(N\left[\mathcal{I}, \mathcal{J}\right]\right)^{2-2\theta} A_{\mathscr{E},\mathfrak{r}}^{1-\eta} \\
&\leq 2CN\mathfrak{b}^{-2\theta} z^2 A_{\mathscr{E},\mathfrak{r}}^{1-\eta} \sum_{N\mathcal{I} \leq z} N\mathcal{I}^3 \\
&\ll 2CN\mathfrak{b}^{-2\theta} z^{3d+3} A_{\mathscr{E},\mathfrak{r}}^{1-\eta} \leq 2CA_{\mathscr{E},\mathfrak{b}\mathfrak{r}},
\end{aligned}$$

where we have used the basis of characters for the finite groups $\left(\mathcal{O}_F / \left[\mathcal{I}, \mathcal{J}\right]\right)^2$ and the remainder term is defined by

$$r_{\mathcal{I}}(y) = |\mathscr{E} \cap (x_0 + y + \mathcal{I}\mathfrak{b}\mathfrak{r})| - \frac{1}{N\mathcal{I}^2} |\mathscr{E} \cap (x_0 + \mathfrak{b}\mathfrak{r})|,$$

which is bounded above using the triangle inequality and Assumption 7.2.3, as follows:

$$\begin{aligned}
\left|r_{[\mathcal{I},\mathcal{J}]}(y)\right| &\leq \left|\#(\mathscr{E}\cap(x_0+y+[\mathcal{I},\mathcal{J}]\,\mathfrak{br})) - \frac{1}{\mathrm{Nm}([\mathcal{I},\mathcal{J}]^2)}A_{\mathscr{E},\mathfrak{br}}\right| \\
&\quad + \frac{1}{\mathrm{Nm}\,[\mathcal{I},\mathcal{J}]^2}\left|\#(\mathscr{E}\cap(x_0+\mathfrak{br})) - A_{\mathscr{E},\mathfrak{br}}\right| \\
&\leq C\left(\mathrm{Nm}\,[\mathcal{I},\mathcal{J}]^{-2\theta} + \mathrm{Nm}\,[\mathcal{I},\mathcal{J}]^{-2}\right)(N\mathfrak{b}^2)^{-\theta}A_{\mathscr{E},\mathfrak{r}}^{1-\eta} \\
&\leq 2C\mathrm{Nm}[\mathcal{I},\mathcal{J}]^{-2\theta}\,(N\mathfrak{b})^{-2\theta}\,A_{\mathscr{E},\mathfrak{r}}^{1-\eta}.
\end{aligned}$$

To do this, we must have the bound that $N\,[\mathcal{I},\mathcal{J}]^2\,N\mathfrak{b}^2 \leq A_{\mathscr{E},\mathfrak{r}}$, which would follow from the assumption that

$$z^4 \leq A_{\mathscr{E},\mathfrak{r}}N\mathfrak{b}^{-2}.$$

Using Assumption 7.2.3, we get that $\Delta \ll_C A_{\mathscr{E},\mathfrak{br}}$.

We can now apply Kowalski's large sieve inequality:

Proposition 7.2.5. *If*

$$1 \leq z \leq \min\left(A_{\mathscr{E},\mathfrak{r}}^{1/4}N\mathfrak{b}^{-2},\,\left(A_{\mathscr{E},\mathfrak{r}}^{\eta}N\mathfrak{b}^{2(\theta-1)}\right)^{1/(3d+3)}\right),$$

then

$$S := |S(\mathscr{E}\cap(x_0+\mathfrak{br}),\Omega;\mathcal{L}^*)| \ll_C A_{\mathscr{E},\mathfrak{br}}\prod_{N\mathfrak{p}\leq z}\left(1-\frac{|\Omega_\mathfrak{p}|}{N\mathfrak{p}^2}\right).$$

Proof. The large sieve inequality [Kow08, Proposition 2.3] shows that

$$|S(\mathscr{E}\cap\mathfrak{r},\Omega;\mathcal{L}^*)| \leq \Delta H^{-1}$$

where

$$H = \sum_{\mathcal{I}\in\mathcal{L}}\prod_{\mathfrak{p}|\mathcal{I}}\frac{|\Omega_\mathfrak{p}|}{N\mathfrak{p}^2-|\Omega_\mathfrak{p}|}.$$

Recall that $\mathcal{L}$ consists of the square-free integral ideals of $\mathcal{O}_F$ with norm $\leq z$. Here, we are assuming that $\Omega_\mathfrak{p} \neq \mathbb{F}_\mathfrak{p}^2$, however if it did for some such $\mathfrak{p}$ then the result would be trivial. We bound H as follows:

$$\prod_{\mathfrak{p}|\mathcal{I}}\frac{|\Omega_\mathfrak{p}|}{N\mathfrak{p}^2-|\Omega_\mathfrak{p}|} \leq \prod_{\mathfrak{p}|\mathcal{I}}\frac{(|\Omega_\mathfrak{p}|/N\mathfrak{p})}{N\mathfrak{p}} = \frac{1}{N\mathcal{I}}\prod_{\mathfrak{p}|\mathcal{I}}\frac{|\Omega_\mathfrak{p}|}{N\mathfrak{p}}.$$

The quantity in the final product is the natural extension of the function $\mathfrak{p}\mapsto|\Omega_\mathfrak{p}|/N\mathfrak{p}$ to the squarefree ideals, and we can assume a completely multiplicative extension of this function to all ideals. Let us call this multiplicative function $g(\mathcal{I})$. Then we are left with giving an upper bound for

$$\sum_{\mathcal{I}\in\mathcal{L}} \frac{g(\mathcal{I})}{N\mathcal{I}}.$$

The function $g(\mathcal{I})$ is bounded above on primes by 2 since the Schwartz-Zippel lemma implies that $|\Omega_{\mathfrak{p}}| \le 2N\mathfrak{p}$.

By [Kha17, Lemma 9.8], which generalises directly to the number field situation, we get

$$H \gg \prod_{N\mathfrak{p}\le z} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right)^{-1}$$

which ends the proof. □

This proposition, as well as a number of corollaries and auxiliary results are used to prove Theorem 7.2.11. We will describe these adjustments below.

Lemma 7.2.6. *We can extend the range of Prop 7.2.5 to any power of $A(\mathscr{E})$ if we allow the constant to depend on that exponent. Let $\varsigma_0 > 0$ which will determine the possible range for z. Suppose that for some $\epsilon > 0$,*

$$\min\left(A_{\mathscr{E},\mathfrak{r}}^{1/4} N\mathfrak{b}^{-1/2}, \left(A_{\mathscr{E},\mathfrak{r}}^{\eta}, N\mathfrak{b}^{2(\theta-1)}\right)^{1/(3d+3)}\right) \ge A_{\mathscr{E},\mathfrak{r}}^{\epsilon}.$$

Then, for any $1 \le z \le (A_{\mathscr{E},\mathfrak{r}})^{\varsigma_0}$,

$$S \ll_{C,\varsigma_0,\epsilon} A_{\mathscr{E},\mathfrak{br}} \prod_{N\mathfrak{p}\le z} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right)$$

Proof. If $z \le A_{\mathscr{E},\mathfrak{r}}^{\epsilon}$, then Prop 7.2.5 applies. Apply the proposition to $z_0 = A_{\mathscr{E},\mathfrak{r}}^{\epsilon}$ to get

$$S \ll_C A_{\mathscr{E},\mathfrak{br}} \prod_{N\mathfrak{p}\le z_0} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right).$$

It suffices to give an upper bound for

$$T = \prod_{z_0 < N\mathfrak{p} < A_{\mathscr{E},\mathfrak{r}}^{\varsigma_0}} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right)^{-1}.$$

Following [Nai92, Lemma 2(i)], we note that up to a uniform additive constant,

$$\begin{aligned}
\log T &= -\sum \log\left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right) \\
&\le 2 \sum_{z_0 < N\mathfrak{p} < A_{\mathscr{E},\mathfrak{r}}^{\varsigma_0}} \frac{1}{N\mathfrak{p}} + O(1) \\
&= 2\left(\log\log A_{\mathscr{E},\mathfrak{r}}^{\varsigma_0} - \log\log A_{\mathscr{E},\mathfrak{r}}^{\epsilon}\right) + O(1) \\
&= 2\log\frac{\varsigma_0}{\epsilon} + O(1)
\end{aligned}$$

The growth of the harmonic sums here are by the same growth for $\mathbb{Q}$ and the Chebotarev Density Theorem, since only the totally split primes contribute significantly to the harmonic sum and these have density $\frac{1}{[F:\mathbb{Q}]}$. $\square$

Lemma 7.2.7. *We can insert a congruence condition on the value of* $Q(x)$. *Pick an ideal* $\mathfrak{a} \subset \mathcal{O}_F$ *(not necessarily square-free), and define*

$$S_{\mathfrak{a}} := |\{x \in S(\mathscr{E} \cap \mathfrak{r}, \Omega; \mathcal{L}_{\mathfrak{a}}^*) : Q(x) \in \mathfrak{a}_f, \gcd(\mathfrak{a}, Q(x)/\mathfrak{a}) = 1\}|$$
$$S'_{\mathfrak{a}} := |\{x \in S(\mathscr{E} \cap \mathfrak{r}, \Omega; \mathcal{L}_{\mathfrak{a}}^*) : Q(x) \in \mathfrak{a}_f\}|$$

Here $\mathcal{L}_{\mathfrak{a}}^*$ *is the prime sieve support* $\mathcal{L}_{\mathfrak{a}}^* = \{\mathfrak{p} \in \Sigma_F : N\mathfrak{p} \leq z, (\mathfrak{a}, \mathfrak{p}) = 1\}$. *Suppose that*

$$N\mathfrak{a}^2 \leq A_{\mathscr{E},\mathfrak{r}}^{\min\left(1/2, \frac{\eta}{2(1-\theta)}\right)},$$

and there exists a positive constant $\varsigma > 0$ *such that*

$$1 \leq z \leq A_{\mathscr{E},\mathfrak{r}}^{\varsigma}.$$

Then

$$S_{\mathfrak{a}} \ll_{C,\eta,\theta,\varsigma} \frac{A_{\mathscr{E},\mathfrak{r}} \widetilde{\rho}_Q(\mathfrak{a})}{N(\mathfrak{a})^2} \prod_{N\mathfrak{p} \leq z, \mathfrak{p} \nmid \mathfrak{a}} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right),$$
$$S'_{\mathfrak{a}} \ll_{C,\eta,\theta,\varsigma} \frac{A_{\mathscr{E},\mathfrak{r}} \rho_Q(\mathfrak{a})}{N(\mathfrak{a})^2} \prod_{N\mathfrak{p} \leq z, \mathfrak{p} \nmid \mathfrak{a}} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right).$$

Proof. Assume there is no $\mathfrak{p} \nmid \mathfrak{a}$ with $N\mathfrak{p} \leq z$ and $Q \equiv 0 \pmod{\mathfrak{p}}$ otherwise the statement is trivial.

Consider a possible solution $x_0 \in \mathfrak{r}/\mathfrak{a}\mathfrak{r}$ such that $Q(x_0)_{\mathfrak{p}} \in \mathfrak{a}_{\mathfrak{p}}, \forall \mathfrak{p}$. Choose uniformisers $a_\nu \in \mathcal{O}_{F_\nu}$ at each place, then we can consider the form

$$Q_0 : x_0 + \mathfrak{a}\mathfrak{r} \longrightarrow \prod_{\nu \nmid \infty} \mathcal{O}_{F_\nu}$$
$$x \longmapsto (a_\nu^{-1} Q(x)_\nu)_\nu,$$

To apply Lemma 7.2.6, we need to check that the conditions of the lemma hold for Q_0. They are satisfied with $\epsilon = \frac{\eta}{2(3d+3)}$ uniformly independent of $\mathfrak{r}, Q, C$. Applying the lemma, and using the fact that for $\mathfrak{p} \nmid \mathfrak{a}$, the value $|\Omega_{\mathfrak{p}}|$ is independent of $\mathfrak{a}$, we get that the contribution, $S_{\mathfrak{a}}(x)$, to $S_{\mathfrak{a}}$ coming from the residue class of x is bounded above by

$$S_{\mathfrak{a}}(x) \ll_{C,\delta,\varsigma} \frac{A_{\mathscr{E},\mathfrak{r}}}{N\mathfrak{a}^2} \prod_{N\mathfrak{p} \leq z, \mathfrak{p} \nmid \mathfrak{a}} \left(1 - \frac{|\Omega_{\mathfrak{p}}|}{N\mathfrak{p}^2}\right)$$

Now we simply need to determine how many such residue classes contribute. Of course, $Q(x_0) \in \mathfrak{a}$ implies that the point (x_0) is one of the $\rho_Q(\mathfrak{a})$ solutions modulo $\mathfrak{a}$. This proves the result for $S'_{\mathfrak{a}}$.

For $S_{\mathfrak{a}}$, the fact that $\forall \mathfrak{p}^k || \mathfrak{a}$, $\mathfrak{p}^k || Q(x)$ implies that $\mathfrak{p} \nmid Q_0$. Therefore, the point $(x_0, y_0)_{\mathfrak{p}} \in \left(\mathcal{O}/\mathfrak{p}^k\right)^2$ does not have $N\mathfrak{p}^2$-lifts, so contributes to $\widetilde{\rho}_Q(\mathfrak{p}^k)$. The result follows. $\square$

In the above result, we can include any prime factor $\mathfrak{p}|\mathfrak{a}$ such that $|\Omega_{\mathfrak{p}}| \neq N\mathfrak{p}^2$ in the product at the expense of multiplying by $\theta_Q(\mathfrak{a})$ which is the value of the multiplicative function defined by

$$\theta_Q(\mathfrak{p}^k) = \begin{cases} 1 + 2\frac{\rho_Q(\mathfrak{p})}{N\mathfrak{p}^2}, & \text{if } \mathfrak{p} \nmid Q \\ 1, & \text{o/w} \end{cases}.$$

This is a simple computation we refer to [Kha19a]. Also, the next three Lemmas are proven exactly as Lemmas 9.20, 9.21 and 9.23.

Lemma 7.2.8. *Suppose that there are $C > 0, 1 \geq r > 0$ satisfying $\widetilde{\rho}_Q(\mathfrak{p}^k) \leq CN\mathfrak{p}^{k(2-r)}$ for all prime powers $\mathfrak{p}^k$. Let f be a non-negative multiplicative function on integral ideals of $\mathcal{O}_F$ such that the induced function on $\prod_{\nu \nmid \infty} \mathcal{O}_{F_\nu}$ is of class $\mathscr{M}(A, B, \epsilon)$. Then for any $z \geq 1$,*

$$\sum_{N\mathfrak{a} \leq z} \frac{\widetilde{\rho}_Q(\mathfrak{a}) f(\mathfrak{a})}{N\mathfrak{a}^2} \theta_Q(\mathfrak{a}) \ll_{B,C,r,\epsilon} \sum_{N\mathfrak{a} \leq z} \frac{\widetilde{\rho}_Q(\mathfrak{a}) f(\mathfrak{a})}{N\mathfrak{a}^2}.$$

Lemma 7.2.9. *Let C, r, f be as above. Then for any $\alpha, s > 0, z > 1$ and $(r - \epsilon)\log(z)/(2s) \geq \kappa > 0$,*

$$\sum_{\substack{z^\alpha \leq N\mathfrak{a} \leq z \\ N\mathfrak{p}^+(\mathfrak{a}) \leq z^{1/s}}} \frac{\widetilde{\rho}_Q(\mathfrak{a}) f(\mathfrak{a})}{N\mathfrak{a}^2} \ll_{\kappa,A,B,\epsilon,C,r} e^{-s\alpha\kappa} \sum_{N\mathfrak{a} \leq z} \frac{\widetilde{\rho}_Q(\mathfrak{a}) f(\mathfrak{a})}{N\mathfrak{a}^2}.$$

Lemma 7.2.10. *Let C, r be as above. Then for any $\beta > 0$ and $1 \geq \alpha \geq 0$,*

$$\sum_{\substack{z^\alpha \leq N\mathfrak{a} \leq z \\ N\mathfrak{p}^+(\mathfrak{a}) \leq \log z \log\log z}} \frac{\widetilde{\rho}_Q(\mathfrak{a})}{N\mathfrak{a}^2} \ll_{C,\alpha,\beta} z^{-r\alpha+\beta}.$$

With these in place, we can now prove Theorem 7.2.11.

Theorem 7.2.11. *Suppose that there are $C > 0, 1 \geq r > 0$ satisfying $\widetilde{\rho}_Q(\mathfrak{p}^k) \leq CN\mathfrak{p}^{k(2-r)}$ for all prime powers $\mathfrak{p}^k$. Let $X \geq 1$ be a constant satisfying*

$$\max\{NQ(x) : x \in \mathscr{E} \cap \mathfrak{r}\} \leq X \leq A^\delta_{\mathscr{E},\mathfrak{r}}$$

for some $\delta > 0$.

Let f *be a non-negative multiplicative function on integral ideals of* $\mathcal{O}_F$ *of class* $\mathscr{M}(A, B, \epsilon)$ *for some* $A \geq 1, B > 0$ *and* $0 < \epsilon < \min\{r, \eta r/(4\delta)\}$. *Then*

$$\sum_{x \in \mathscr{E} \cap \mathfrak{r}} f(Q(x)) \ll A(\mathscr{E}) \prod_{N\mathfrak{p} \leq X, \mathfrak{p} \nmid Q} \left(1 - \frac{\rho_Q(\mathfrak{p})}{\mathfrak{p}^2}\right) \sum_{N\mathfrak{a} \leq X} \frac{f(\mathfrak{a})\widetilde{\rho}_Q(\mathfrak{a})}{\mathfrak{a}^2}$$

where the implicit constant depends only on $C, A, B, \epsilon, \delta$.

Proof. Let $Z = (A(\mathscr{E}))^{\eta/3}$. For any element $(x, y) \in \mathscr{E} \cap \mathcal{O}_F^2$, we decompose the ideal $(Q(x, y)) \subset \mathcal{O}_F$ into prime ideals. If we take the norm of this, we get a decomposition into rational primes,

$$NQ(x, y) = p_1^{e_1} ... p_l^{e_l}.$$

As in the rational case, we split this up into a product ab with $a = p_1^{e_1} ... p_j^{e_j}$ chosen such that $a \leq Z$ but $ap_{j+1}^{e_{j+1}} > Z$. Correspondingly, we get a decomposition

$$(Q(x, y)) = \mathfrak{a}\mathfrak{b}$$

of ideals in $\mathcal{O}_F$ with $N\mathfrak{a} = a, N\mathfrak{b} = b$. Let $q \in \mathbb{N}$ denote the minimum norm of a prime divisor of $\mathfrak{b}$. We will treat here just the leading term contribution towards the bound of Theorem 7.2.11. This corresponds to the points

$$R_1 = \left\{(x, y) \in \mathscr{E} \cap \mathcal{O}_F^2 : q \geq Z^{1/2}\right\}$$

We wish to compute

$$\sum_{(x,y) \in R_1} f(Q(x, y)).$$

Note that since $\mathfrak{a}, \mathfrak{b}$ are coprime, $f(Q(x, y)) = f(\mathfrak{a})f(\mathfrak{b})$. Also, $Z^{\frac{1}{2}\Omega(\mathfrak{b})} \leq N\mathfrak{b} \leq X \leq A(\mathscr{E})^\delta$, and so $\Omega(\mathfrak{b}) \ll 1$ and by the definition of $f \in \mathscr{M}(A, B, \epsilon)$, $f(\mathfrak{b}) \ll 1$. Therefore

$$\sum_{R_1} f(Q(x, y)) \ll \sum_{N\mathfrak{a} \leq Z} f(\mathfrak{a}) S_\mathfrak{a}$$

(here, we are using $\mathfrak{a}$ in the same capacity as $\mathfrak{b}$ was used in Lemma 7.2.7, since now we are considering the case of $\mathscr{E} \cap \mathcal{O}_F^2$). By Lemma 7.2.7 applied with the data

$$(\mathfrak{a}, \mathfrak{b}, z, \delta, \eta, \varsigma) = \left(\mathcal{O}_F, \mathfrak{a}, Z^{1/2}, \eta/3, \eta, \eta/6\right),$$

we get

$$\sum_{R_1} f(Q(x, y)) \ll A(\mathscr{E}) \prod_{\substack{\deg(Q) < N\mathfrak{p} \leq Z^{1/2} \\ \mathfrak{p} | Q}} \left(1 - \frac{|\Omega_\mathfrak{p}|}{N\mathfrak{p}^2}\right) \sum_{N\mathfrak{a} \leq Z} \frac{\widetilde{\rho}_Q(\mathfrak{a}) f(\mathfrak{a})}{N\mathfrak{a}^2} \theta_Q(\mathfrak{a})$$

The sums over R_2, R_3, R_4 are left out as they are again similar to [Kha19a] Theorem 9.7. □

Chapter 8

Final Proofs

In this short final chapter, we finally bring together the components of the previous chapters. The method at this point is exactly as in the papers of Khayutin ([Kha17] and [Kha19b]), so we may be brief.

8.1 Kuga-Sato Case

Since the Mixing Conjecture (Theorem 2.6.5 modelled on Conjecture 2 of [MV06]) is the main focus of this thesis, here we will simply sketch how to combine the results of the previous chapters to deduce a Kuga-Sato equidistribution result (Theorem 2.6.4) over totally real fields.

We apply Theorem 3.6.1 via a contradiction. The ergodic decomposition in this theorem is understood in the Kuga-Sato case using Theorem 3.3.9, which says that the ergodic decomposition will consist of algebraic measures on the subsets of the form $[\mathbb{P}(\mathbb{A})^+\xi]$ or $[\mathbb{G}(\mathbb{A})^+\xi]$. To deduce invariance under $\mathbb{P}(\mathbb{A})^+$, we need to rule out the second type. Therefore, we assume by contradiction, that we can find a compact subset $C_{\mathcal{M}}$ consisting of measures of the second form (in particular, we can set $\mathcal{I}$ to simply be the singleton $\{h_{\mathbb{G}^{sc}}(a)\}$).

To contradict the conclusion of Theorem 3.6.1, we must uniformly bound the correlations between the toral measures μ_i and the measures λ associated to $[\mathbb{G}(\mathbb{A})^+\xi]$, in the sense of Theorem 3.6.1 (note that the Theorem guarantees that no such $A, \epsilon > 0$ can exist).

To bound this correlation, we apply Proposition 6.1.3 which gives the geometric expansion of the correlation as

$$\mathrm{Corr}(\mu_i, \nu)\left[B^{(-n,n)}\right] = \sum_{\substack{0 \neq v \in T(F)\backslash V(F) \\ \varkappa \in \mathbb{G}(\mathbb{Q})\backslash \mathbb{G}(\mathbb{Q})^+}} \mathrm{RO}_{v,\varkappa}\left(B^{(-n,n)}\right)$$

where

$$\mathrm{RO}_{v,\varkappa}(B^{(-n,n)}) = \int_{\mathbb{G}(\mathbb{A})^+\times\mathcal{T}} 1_{(h,y)^{-1}B^{(-n,n)}(l,x)}(g^{-1}\varkappa t, g^{-1}v)d(g,t).$$

We may unfold this summation and integral, and use an invariant map similar to the one constructed in Definition 6.1.12 (see Section 5 of [Kha19b]), to get a count of the number of integral ideals of E with bounded norm, which lie in the principal genus and the subgroup defined by $\mathcal{T}$ and are divisible by $\mathfrak{P}^n$.

Finally, such a count of ideals can be expressed as the Mellin transform of a Hecke L-function via Perron's formula. These Hecke L-functions are in fact L-functions of GL_2 modular forms by constructing the associated θ-function, and the bound on the correlation required for Theorem 3.6.1 follows from the subconvex bound of Duke, Friedlander and Iwaniec ([DFI02]). In particular, the achieved bound is that

$$\mathrm{Corr}(\mu_i,\lambda)[B^{(-n,n)}] \leq A\exp\left(6n\sum_{\alpha_\nu\in\Phi^+_{\mathrm{Lyp}}}\alpha_\nu(a)\right).$$

By the leafwise computation of the entropy of the Haar measure (see Proposition 3.5.1) on $\mathbb{G}$, we see that $h_a(\lambda) \leq -2\sum_{\alpha_\nu\in\Phi^+_{\mathrm{Lyp}}}\alpha_\nu(a)$, and so Theorem 3.6.1 applies to show that no diagonal measures contribute to the ergodic decomposition. Therefore μ is $\mathbb{P}(\mathbb{A})^+$-invariant, proving Theorem 2.6.4.

8.2 The Mixing Conjecture

In this section, we prove mixing for quaternion algebras over totally real fields. We first need a Proposition from [Kha17] (Proposition 10.15 of that paper, which is proven via Section 10.3).

Proposition 8.2.1. *Assume the following*

1. *$C > 0$ satisfies*

$$\frac{L'(1,\chi_E)}{L(1,\chi_E)} \leq C\log|D_E|,$$

2. *For χ_j running over the characters of $\mathbb{A}_E^\times$ that are trivial on $\mathcal{T}$ (all of which are quadratic),*

$$\sum_{j:\chi_j\neq 1} L(1,\chi_j) \ll L(1,\chi_E)\log|D_E|.$$

3. There exists $\eta > 0$ such that

$$\mathfrak{R} := \min_{\substack{\mathfrak{a} \subset \Lambda \\ [\mathfrak{a}] = [\mathfrak{s}]}} N\mathfrak{a} \geq |D_\Lambda|^{\frac{1}{2(2d+1)} + \eta},$$

and Assumption 7.1.1 holds (giving bounds for the Gauss sums of equation (7.3)), i.e. there is a $\theta > 0$ such that

$$|G_{w,a,\mathcal{I},\mathfrak{r}}| \leq (N\mathcal{I}^2)^{1-\theta}.$$

Then there is a continuous function $c : \mathbb{G}(\mathbb{A}) \to \mathbb{R}_{>0}$ depending on the conductor of Λ such that the quantity

$$2^{[F:\mathbb{Q}]} \sum_{\substack{a \in \mathfrak{c}\mathfrak{s}^{-1}\widehat{\Lambda} \\ |a_\nu|_\nu \leq 2^8 |s_\nu|_\nu^{-1} e^\eta \rho \\ \upsilon \mathfrak{P}^{2n} | (\mathrm{Nm}(a) - \zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2} D_{\Lambda/F}}} (f_0 \cdot r_0) \left(\frac{(\mathrm{Nm}(a) - \zeta)(\mathrm{Nm}\mathfrak{s})\mathfrak{c}^{-2} D_{\Lambda/F}}{\upsilon \mathfrak{P}^{2n}} \right),$$

of Proposition 6.1.22 is bounded above by

$$c(\mathrm{ctr}(\xi)) \frac{\sqrt{|D_E|} \left(L(1, \chi_E) + |D_E|^{-\epsilon_0 + o(1)} \right)}{[\mathbb{T}(\mathbb{A}) : \mathcal{T}] N \mathfrak{P}^{2n}}.$$

Recall from the discussion following Corollary 7.1.7 that the assumption given for $\mathfrak{R}$ implies the same assumption for each $\mathfrak{R}_\mathfrak{c}$ for $\mathfrak{c} \in \mathrm{Cl}_F$.

Proof. The proof of this Proposition is exactly as in Section 10.3 of [Kha17], by applying the sieve result of Theorem 7.2.11 to the sum in Proposition 6.1.22, except for a few minor changes which we will indicate now. Firstly, the necessary claims on quadratic forms are contained in Appendix B. Any variation of substance appears in the proof of Proposition 10.14.

The proof of Proposition 10.14 in [Kha17] proceeds via estimation of the quantity

$$\sum_{\substack{N\mathfrak{a} \leq A|D| \\ \gcd(N\mathfrak{a}, m) = 1}} \frac{f(\mathfrak{a})}{N\mathfrak{a}}.$$

The product over primes preceding this logarithmic sum in the statement of Proposition 10.14 of *loc. cit.* is uniformly bounded (except for varying the conductor $\mathfrak{f}_\Lambda$) by $(\log |D|)^{-1}$. Also the congruence condition in the logarithmic sum only contributes terms bounded in terms of the conductor, and so we can ignore it. Recall that f counts the number of integral ideals of Λ whose class lies in the subgroup $\mathcal{T}$ of the class group, with norm $\mathfrak{a}$. The sum is computed via Perron's formula applied to the Dirichlet series

$$L_{\mathcal{T}}(s) = [\mathbb{T}(\mathbb{A}) : \mathcal{T}]^{-1} \sum_j L(\chi_j, s),$$

where χ_j runs over the characters of $\mathbb{A}_E^\times$ vanishing on $\mathcal{T}$. Let $\zeta_E(s) = L(s, \chi_E)\zeta_F(s)$ be the decomposition of the trivial character on E into two characters on F. We get that

$$[\mathbb{T}(\mathbb{A}) : \mathcal{T}]L_{\mathcal{T}}(s) = \frac{L(1, \chi_E)}{s-1} + \left(L(\chi_E, 1)\gamma_F + L'(1, \chi_E) + \sum_{j:\chi_j \neq 1} L(1, \chi_j) \right) + O(s-1).$$

Perron's formula tells us that for $c > 0$,

$$\sum \frac{f(\mathfrak{a})}{N\mathfrak{a}} \varphi\left(\frac{N\mathfrak{a}}{A|D|}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L_{\mathcal{T}}(s+1) \left(\mathcal{M}\varphi(s)(A|D|)^s\right) ds,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a smooth function (approximating the indicator function $1_{[0,1]}$) as described in [Kha17] (Proposition 10.14) and $\mathcal{M}$ denotes the Mellin transform. Therefore,

$$[\mathbb{T}(\mathbb{A}) : \mathcal{T}] \sum \frac{f(\mathfrak{a})}{N\mathfrak{a}} \leq \left(\sum_{j:\chi_j \neq 1} L(1, \chi_j) + L(1, \chi_E) \left(\log(A|D|) + \gamma_F + \frac{L'(1, \chi_E)}{L(1, \chi_E)} + \int_1^\infty \frac{\varphi(x)}{x} dx \right) \right)$$
$$+ \frac{1}{2\pi} \sum_j \int_{-\infty}^{\infty} \frac{|L(1/2 + it, \chi_j)|}{(A|D|)^{1/2}} |\mathcal{M}\varphi(-1/2 + it)| dt$$

We then require any strength of subconvexity result

$$|L(1/2 + it, \chi_j)| \ll |D|^{1/2 - \epsilon_0 + o(1)} |1/2 + it|^B,$$

which is given by Theorem 6 of [MV06]. Along with the assumptions of the Proposition on $\sum_{\chi_j \neq 1} L(1, \chi_j)$, we deduce the result as required. □

We now combine this Proposition with previous results to deduce our main result. Before we do this, for reference we list here all of our set-up and assumptions once more.

We have a quaternion algebra B defined over a totally real field, F, (with $[F : \mathbb{Q}] = d$) from which we defined the algebraic group $G = PB^\times$ of projective units in the quaternion algebra, and we take a finite set, S, of finite places of $\mathbb{Q}$ such that F is completely split at all places of S, B split at all places above S, and $|S|d > 1$. We have fixed a maximal compact torus $\mathcal{K}_\infty < G(F_\infty)$. We have a sequence of F-algebra embeddings $E_i \to B$ of quadratic CM extensions E_i/F into B, which induce maximal rank anisotropic tori $T_i < G$. For each torus T_i, we have $g_i \in G(\mathbb{A}_F)$, $s_i \in T_i(\mathbb{A}_F)$, and a finite index subgroup $\mathcal{T}_i < T_i(\mathbb{A}_F)$. To these, we associate the homogeneous joint toral set

$$[\mathcal{T}_i(g_i, s_i g_i)] \subset (G(F) \setminus G(\mathbb{A}_F))^2 .$$

We assume these are $\mathcal{K}_\infty$-invariant, in the sense that $g_i^{-1} T_i(F_\infty) g_i = \mathcal{K}_\infty$.

We can associate to these toral sets an order $\Lambda_i \subset E_i$ with discriminant D_i and conductor $\mathfrak{f}_i$, as in 2.3.1. We assume the following conditions on the subgroup $\mathcal{T}$:

1. $\mathcal{T}_i = T_i(F_\infty) \prod_\nu \mathcal{T}_{i,\nu}$ splits as a product over the places of F. In addition, assume that $\mathcal{T}_{i,\nu} = T_i(F_\nu)$ at all places ν where B is ramified.

2. $\mathcal{T}_i$ corresponds to a subgroup of the class group

$$T_i(F) \setminus T_i(\mathbb{A}_F^\infty)/(T_i(\mathbb{A}_F^\infty) \cap g_{i,f} \mathcal{K}_f g_{i,f}^{-1})$$

(which is related to the class group of the order Λ_i constructed in 2.3.1). In particular $T_i(F) \left(T_i(\mathbb{A}_F^\infty) \cap g_{i,f} \mathcal{K}_f g_{i,f}^{-1}\right) < \mathcal{T}_i$.

3. $\mathcal{T}_i$ contains the intersection $T_i(\mathbb{A}_F) \cap G(\mathbb{A}_F)^+ = \mathrm{im}(B^{(1)}(\mathbb{A}_F) \to G(\mathbb{A}_F))$, where $B^{(1)}$ is the algebraic group of norm 1 elements of B.

4. $\mathcal{T}_i$ is preserved by the Galois action of $\mathrm{Gal}\,(E/F)$.

To such a homogeneous toral set we associated, in Definition 2.5.2, a discriminant, $\mathrm{disc}([\mathcal{T}_i(g_i)]) = \prod_p \mathrm{disc}([\mathcal{T}_{i,p}(g_{i,p})])$, which by Proposition 2.5.3 is essentially the discriminant, D_i, of the associated order Λ_i.

We write $(PB^\times \times PB^\times)(\mathbb{A}_F)^+$ for the image of $\left(B^{(1)} \times B^{(1)}\right)(\mathbb{A}_F)$ under the natural projection map (see Definition 3.3.7 and Proposition 3.3.8 which shows that this notation agree with the unipotents definition).

Also, we defined

$$\mathfrak{R}_i = \min_{\substack{\mathfrak{a} \subset \Lambda_i \\ [\mathfrak{a}] = [\mathfrak{s}_i]}} N\mathfrak{a},$$

and Gauss sums $G_{w,a,\mathcal{I},\mathfrak{r}}$ as in equation (7.3) of Section 7.1.

Theorem 8.2.2. *Let μ_i be a sequence of probability measures on $[(PB^\times \times PB^\times)(\mathbb{A}_F)]$ associated to homogeneous joint toral sets $[\mathcal{T}_i(g_i, s_i g_i)]$. Assume the following*

1. *The discriminants $|D_i| \to \infty$, and the tori $\mathbb{T}_i$ are split at places above S.*

2. *The local discriminants at S are uniformly bounded, i.e.*

$$\forall p \in S, \mathrm{disc}_p([\mathcal{T}_i g_i]) \ll 1$$

3. *The conductors are bounded*

$$\mathfrak{f}_i \ll 1$$

4. *There exists $C > 0$ satisfying*

$$\frac{L'(1, \chi_{E_i})}{L(1, \chi_{E_i})} \leq C \log |D_{E_i}|,$$

5. *For $\chi_j^{(i)}$ running over the characters of $\mathbb{A}_{E_i}^\times$ that are trivial on $\mathcal{T}_i$ (all of which are quadratic),*

$$\sum_{j:\chi_j^{(i)} \neq 1} L(1, \chi_j^{(i)}) \ll L(1, \chi_{E_i}) \log |D_{E_i}|, \text{ as } i \to \infty.$$

6. *There exists $\eta > 0$ such that*

$$\mathfrak{N}_i \geq |D_i|^{\frac{1}{2(2d+1)}+\eta},$$

and $\theta > 0$ such that

$$|G_{w,a,\mathcal{I},\mathfrak{r}}| \leq (N\mathcal{I}^2)^{1-\theta}.$$

Then any weak- limit of the sequence $\{\mu_i\}$ is $(PB^\times \times PB^\times)(\mathbb{A}_F)^+$-invariant.*

Proof. Firstly, by applying Proposition 3.6.3 using the assumption of bounded discriminants at the places $p \in S$, we may assume that the measures μ_i are in fact A^+-invariant. Let μ be some weak-* limit of the sequence μ_i. We now wish to apply Theorem 3.6.1 which says that over a positive measure (using the measure from the ergodic decomposition of μ) compact subset of $\mathcal{M}$, the space of A^+-invariant and ergodic probability measures, there can be no upper bound on the correlation of μ_i with an algebraic measure in that subset stronger than that determined by the entropy . The ergodic decomposition in this case is determined by Theorem 3.3.13, with intermediate measures as in Proposition 3.4.5 (strictness of the limit and $\mathbb{G}(\mathbb{A})^+$-invariance of the projections is simply the statement of single equidistribution, see e.g. Theorem 4.6 of [Ein+07]).

Suppose (aiming for a contradiction) that a non-trivial part of the ergodic decomposition of μ is from algebraic measures supported on sets of the form $\left[\mathbb{G}^\Delta(\mathbb{A})^+\xi\right]$.

That is, we take a compact subset $C_{\mathcal{M}}$ consisting of these measures with $\mathscr{P}(C_{\mathcal{M}}) > 0$. Theorem 3.6.1 tells us that for regular $a \in A^+$, there can then be no bound on the correlations $\mathrm{Corr}(\mu_i, \nu)[B^{(-n,n)}]$ stronger than $e^{-2nh_{\mathbb{G}^{sc}}(a)}$. We will, however, construct such a bound.

By the assumption that $|D_i| \to \infty$, we see that $\mathfrak{R}_i \to \infty$ and therefore Assumption 6.1.5 required for Lemma 6.1.6 holds for $i \gg 1$ (which removes the non-stable parts of the geometric expansion, i.e. those with non-compact stabilisers), so the expansion of Theorem 6.1.21 is valid. Therefore, to deduce a stronger bound on the correlation, we require that the sum in Proposition 6.1.22 is bounded by $c(\mathrm{ctr}(\xi))\mathrm{vol}([\mathcal{T}g])e^{-\epsilon n h_{\mathbb{G}^{sc}}(a)}$ for some $\epsilon > 0$.

The sum over multiplicative functions of Proposition 6.1.22 is handled in Proposition 8.2.1 using the analytic input of Section 7. Inputting this into the geometric expansion of Theorem 6.1.21 we deduce that, for $\mathrm{ctr}(\xi)$ in the compact subset $C_{\mathcal{M}}$,

$$\mathrm{Corr}(\mu_i, \nu)[B^{(-n,n)}] \ll_{F, C_{\mathcal{M}}} e^{-4nh_{\mathbb{G}^{sc}}(a)} \frac{\sqrt{|D_{E_i}|}\left(L(1, \chi_{E_i}) + |D_{E_i}|^{-\epsilon_0 + o(1)}\right)}{[\mathbb{T}_i(\mathbb{A}) : \mathcal{T}_i]\mathrm{vol}([\mathcal{T}_i g_i])}.$$

Here we have used that by Lemma 4.2.3, the $\mathbb{G}^{\Delta}(\mathbb{A})$-volume term of Theorem 6.1.21 is bounded for $\mathrm{ctr}(\xi) \in C_{\mathcal{M}}$. Finally, we apply the toral volume computation of Lemma 4.2.1, noting that under our assumptions on $\mathcal{T}$, the inertia $f_{\mathcal{L}}(\mathcal{T}_i) = [\mathbb{T}_i(\mathbb{A}_F) : \mathcal{T}_i]^{-1}$, to deduce that

$$\mathrm{Corr}(\mu_i, \nu)\left[B^{(-n,n)}\right] \leq Ce^{-4nh_{\mathbb{G}^{sc}}(a)} + o(1) \text{ as } i \to \infty.$$

Thus we have the stronger bound on the decay of the correlation than is allowed under Theorem 3.6.1. Consequently, no algebraic measures can contribute to the ergodic decomposition of μ other than those of the form $[(\mathbb{G} \times \mathbb{G})(\mathbb{A})^+\xi]$, and we finally arrive at the desired result, that μ is invariant under $(\mathbb{G} \times \mathbb{G})(\mathbb{A})^+$. □

Appendix A

Principal Genus Theory

This appendix works through the totally real case of principal genus theory in analogue with Appendix A of [Kha19a]. When the proofs are identical, we do not write them out.

Let Λ be an $\mathcal{O}_F$-order in a CM field K over a totally real field F. There is a short exact sequence

$$1 \to \mathrm{Pic}(\Lambda)/\mathrm{Pic}(\Lambda)^2 \xrightarrow{\mathrm{Nm}} F^\times \setminus \mathbb{A}_F^\times / F_\infty^{\gg 0} \prod_{\nu \nmid \infty} \mathrm{Nr}\Lambda_\nu^\times \xrightarrow{\chi_K} \{\pm 1\} \to 1$$

where χ_K is the adelic character attached to K/F by class field theory, and $F_\infty^{\gg 0}$ is the set of totally positive elements of $F_\infty^\times$.

For $\Lambda_\nu \leq \mathcal{O}_{K_\nu}$, either

- $\Lambda_\nu = \mathcal{O}_{K_\nu}$ and so $\Lambda_\nu^\times = \mathcal{O}_{K_\nu}^\times$. In this case, $\mathrm{Nm}\Lambda_\nu^\times$ is equal to $\mathcal{O}_{F_\nu}^\times$ when K/F is unramified, and is equal to an index 2 subgroup otherwise. When the characteristic is prime to 2, the only index 2 subgroup is $\mathrm{Nm}\Lambda_\nu^\times = \mathrm{Nm}\mathcal{O}_{K_\nu}^\times = \left(\mathcal{O}_{F_\nu}^\times\right)^2$. If the characteristic is equal to 2 and K_ν/F_ν is ramified, there are $2^{2^{[F_\nu:\mathbb{Q}_2]}} - 1$ possible subgroups; or

- $\Lambda_\nu = \mathcal{O}_{F_\nu} + \mathfrak{f}_\nu \mathcal{O}_{K_\nu}$, and $\Lambda_\nu^\times = \mathcal{O}_{F_\nu}^\times + \mathfrak{f}_\nu \mathcal{O}_{K_\nu}$ for a non-trivial ideal $\mathfrak{f}_\nu \subset \mathcal{O}_{F_\nu}$. In this case, the options for the norm group are
  - If the characteristic is prime to 2, then
  $$\mathrm{Nr}\Lambda_\nu^\times = (\mathcal{O}_{F,\nu}^\times)^2$$
  - If the characteristic is equal to 2, then
  $$\left[\mathcal{O}_{F,\nu}^\times : \mathrm{Nm}\Lambda_\nu^\times\right] \text{ divides } 2^{[F_\nu:\mathbb{Q}_2]}.$$

Lemma A.1.

$$\#\text{Pic}(\Lambda)[2] | 2^{[F:\mathbb{Q}]+\mu_{tame}+1} h_F^+$$

where μ_{tame} is the number of places not dividing 2 where Λ ramifies, i.e. those dividing the relative different $\mathfrak{d}_{\Lambda/F}$.

Proof. By the short exact sequence above, we get that

$$\#\text{Pic}(\Lambda)[2] = 2h_F^+ \left| \prod_{\nu \nmid \infty} \mathcal{O}_{F_\nu}^\times / \left(\prod_{\nu \nmid \infty} \mathcal{O}_{F_\nu}^\times \right) \cap \left(F^\times F_\infty^{\gg 0} \prod_{\nu \nmid \infty} \text{Nr}\Lambda_\nu^\times \right) \right|$$

Therefore,

$$\#\text{Pic}(\Lambda)[2] \text{ divides } 2h_F^+ \prod_{\nu \nmid \infty} \left| \mathcal{O}_{F_\nu}^\times / \text{Nr}\Lambda_\nu^\times \right|$$

and the result follows. □

Proposition A.2. *There is a cohomological interpretation of* Pic(Λ)[2]. *In fact* $\text{Pic}(\Lambda)[2] = \text{Pic}(\Lambda)^{\mathfrak{G}}$ *and*

$$\frac{h_F}{2} \prod_{\nu \nmid \infty} |H^1(\mathfrak{G}, \Lambda_\nu^\times)| \leq |\text{Pic}(\Lambda)[2]| \leq 2^{[F:\mathbb{Q}]} h_F \prod_{\nu \nmid \infty} |H^1(\mathfrak{G}, \Lambda_\nu^\times)|.$$

Proof. Consider

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda^\times & \longrightarrow & K^\times & \longrightarrow & \mathscr{P}(\Lambda) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \prod_{\nu \nmid \infty} \Lambda_\nu^\times & \longrightarrow & \prod'_{\nu \nmid \infty} K_\nu^\times & \longrightarrow & \mathscr{J}(\Lambda) & \longrightarrow & 1 \end{array}$$

Taking $\mathfrak{G} = \text{Gal}(K/F)$-cohomology,

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & \mathcal{O}_F^\times & \longrightarrow & F^\times & \longrightarrow & \mathscr{P}(\Lambda)^{\mathfrak{G}} & \longrightarrow & H^1(\mathfrak{G}, \Lambda^\times) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \prod_{\nu \nmid \infty} \mathcal{O}_{F_\nu}^\times & \longrightarrow & \prod'_{\nu \nmid \infty} F_\nu^\times & \longrightarrow & \mathscr{J}(\Lambda)^{\mathfrak{G}} & \longrightarrow & \prod_{\nu \nmid \infty} H^1(\mathfrak{G}, \Lambda_\nu^\times) & \longrightarrow & 1 \end{array}$$

The last terms are trivial due to Hilbert 90. This becomes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{P}(\mathcal{O}_F) & \longrightarrow & \mathscr{P}(\Lambda)^{\mathfrak{G}} & \longrightarrow & H^1(\mathfrak{G}, \Lambda^\times) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 1 & \longrightarrow & \mathscr{J}(\mathcal{O}_F) & \longrightarrow & \mathscr{J}(\Lambda)^{\mathfrak{G}} & \longrightarrow & \prod_{\nu \nmid \infty} H^1(\mathfrak{G}, \Lambda_\nu^\times) & \longrightarrow & 1 \end{array}$$

Now the proof differs from the rational case. The snake lemma tells us that there is an exact sequence

$$1 \to \ker(f) \to \mathrm{Cl}_F \to \mathscr{J}(\Lambda)^{\mathfrak{G}}/\mathscr{P}(\Lambda)^{\mathfrak{G}} \to \mathrm{coker}(f) \to 1$$

To compute the quotient in this sequence, we use the long exact sequence

$$1 \to \mathscr{J}(\Lambda)^{\mathfrak{G}}/\mathscr{P}(\Lambda)^{\mathfrak{G}} \to \mathrm{Pic}(\Lambda)^{\mathfrak{G}} \to H^1(\mathfrak{G}, \mathscr{P}(\Lambda))$$

The right hand term has size at most $2^{[F:\mathbb{Q}]}$ since we have

$$1 \to H^1(\mathfrak{G}, \mathscr{P}(\Lambda)) \to \mathcal{O}_F^\times/\mathrm{Nm}\Lambda^\times \to F^\times/\mathrm{Nm}K^\times$$

and the middle term injects into $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$ which has size $2^{[F:\mathbb{Q}]}$. Therefore

$$2^{-[F:\mathbb{Q}]}\left|\mathrm{Pic}(\Lambda)^{\mathfrak{G}}\right| \leq \left|\mathscr{J}(\Lambda)^{\mathfrak{G}}/\mathscr{P}(\Lambda)^{\mathfrak{G}}\right| \leq \left|\mathrm{Pic}(\Lambda)^{\mathfrak{G}}\right|.$$

Thus we see that

$$\prod_{\nu\nmid\infty} H^1(\mathfrak{G}, \Lambda_\nu^\times) = \frac{|H^1(\mathfrak{G}, \Lambda^\times)|\left|\mathscr{J}(\Lambda)^{\mathfrak{G}}/\mathscr{P}(\Lambda)^{\mathfrak{G}}\right|}{h_F} \leq \frac{2}{h_F}\left|\mathrm{Pic}(\Lambda)[2]\right|.$$

The fact that $|H^1(\mathfrak{G}, \Lambda^\times)| \leq 2$ comes from the simple injection

$$H^1(\mathfrak{G}, \Lambda^\times) \hookrightarrow \mu_\Lambda/\mu_\Lambda^2.$$

□

We can compute the local terms in the product $\prod_{\nu\nmid\infty} H^1(\mathfrak{G}, \Lambda_\nu^\times)$ as follows - the proof being the same as Lemma A.14 in [Kha19a].

Proposition A.3. *If ν is coprime to the discriminant of Λ over $\mathcal{O}_F$,*

$$H^1(\mathfrak{G}, \Lambda_\nu^\times) = 1$$

Otherwise, if ν is coprime to 2,

$$H^1(\mathfrak{G}, \Lambda_\nu^\times) \cong \mathbb{Z}/2\mathbb{Z},$$

with the non-trivial class being represented by $-1 \in \Lambda_\nu^\times$. Finally, if $\nu|2$ and divides the discriminant,

$$\#H^1(\mathfrak{G}, \Lambda_\nu^\times)\,\Big|\,2^{[F_\nu:\mathbb{Q}_2]}$$

Appendix B

Binary Quadratic Forms over Local Fields

For the analysis of the correlation, we need an understanding of binary quadratic forms over the local completions of the totally real field. In this section only, let F denote an arbitrary non-archimedean local field of mixed characteristic $(0, p)$. First, we give a classification of binary quadratic forms over F, and then the possible $\mathcal{O}_F$-lattices.

Let q be an integral primitive binary quadratic form on $\mathcal{O}_F^2 \subset F^2 = V$. Say

$$q(x, y) = \alpha x^2 + \beta xy + \gamma y^2,$$

then we associate to q the discriminant $D_q := \beta^2 - 4\alpha\gamma \in (\mathcal{O}_F \setminus \{0\}) / (\mathcal{O}_F^\times)^2$.[1]

By completing the square, we can choose a basis of V (not necessarily of $\mathcal{O}_F^2$) such that

$$q(x, y) = \alpha_1 x^2 + \alpha_2 y^2.$$

Then $D_q \equiv -4\alpha_1\alpha_2(F^\times)^2 \in F^\times/(F^\times)^2$, and we associate the Hasse symbol

$$S_q = \left(\frac{\alpha_1, -1}{F}\right)\left(\frac{\alpha_2, -D_q}{F}\right)$$

where $\left(\frac{\alpha,\beta}{F}\right)$ is the Hilbert symbol which evaluates to 1 if the equation $\alpha\xi^2 + \beta\eta^2 = 1$ is solvable in F, and to -1 otherwise.

We also consider a non-homogeneous integral binary quadratic of the form

$$Q(x, y) = q(x, y) - \lambda D_Q$$

for $\lambda \in \mathcal{O}_F$. If we just consider equivalence over F (not over $\mathcal{O}_F$), the

[1]Note our convention is the 'two's out' convention, where we suppose the quadratic form is integral rather than the underlying bilinear form.

Proposition B.0.1 ([OMe63], §63C)**.** *The isomorphism classes of (non-degenerate) quadratic forms are determined by their discriminant and Hasse invariant. Furthermore, every such pair* $(D(F^\times)^\times, S) \in F^\times/(F^\times)^2 \times \{\pm 1\}$ *can be realised with the exception of* $\left(1, -\left(\frac{-1,-1}{F}\right)\right)$.

Recall also that we have defined

$$\rho_Q(\mathfrak{p}^k) = \#\left\{(x,y) \in \left(\mathcal{O}_F/\mathfrak{p}^k\right)^2 : Q(x,y) \equiv 0 \pmod{\mathfrak{p}^k}\right\}$$

And $\widetilde{\rho}_Q$ to be the same where we only count points with less that $N\mathfrak{p}^2$ lifts modulo $\mathfrak{p}^{k+1}$ (i.e. exactly $N\mathfrak{p}$ lifts, or none).

B.1 Residue Characteristic Prime to 2

In this section, we cover the necessary results for non-dyadic local fields. Throughout, we assume $p \neq 2$.

Proposition B.1.1 ([OMe63], §92)**.** *Every binary quadratic form on* $\mathcal{O}_F^2$ *with* $q(\mathcal{O}_F^2)\mathcal{O}_F = \mathcal{O}_F$ *and discriminant* $D_q \in \mathcal{O}_F \setminus \{0\}$*, is equivalent to one of the form*

$$q(x,y) = \alpha x^2 + \beta y^2, \alpha \in \mathcal{O}_F^\times, \beta \in \mathcal{O}_F, -4\alpha\beta = D_q.$$

Furthermore,

- *If* $D_q \in \mathcal{O}_F^\times$ *then in fact this is equivalent to* $q(x,y) = x^2 - \frac{1}{4}D_q y^2$*. The equivalence classes of such lattices are determined by* $D_q \in \left(\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2\right)$*, which has size 2.*
- *If* $D_q \in \mathfrak{p}$*, the equivalent class of lattices of this form is determined precisely by* $(\alpha, \beta) \in \left(\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2\right) \times \left((\mathfrak{p} \setminus \{0\})/(\mathcal{O}_F^\times)^2\right)$.

Proposition B.1.2. *Let* $r = \mathrm{ord}_{\mathfrak{p}}(D_q)$ *and* $s = \mathrm{ord}_{\mathfrak{p}}(\lambda)$*. Then if* $r = s = 0$*,*

$$\rho_Q(\mathfrak{p}^k) = N\mathfrak{p}^{k-1}\left(N\mathfrak{p} - \left(\frac{D_q}{\mathfrak{p}}\right)\right).$$

If $r = 0, s \geq 0, k \leq s$*,*

$$\rho_Q(\mathfrak{p}^k) = \left\lceil \frac{k}{2} \right\rceil N\mathfrak{p}^{k-1}\left(1 - \left(\frac{D_q}{\mathfrak{p}}\right)\right) + N\mathfrak{p}^{2\lfloor k/2 \rfloor}.$$

If $r = 0, s \geq 0, k > s$*,*

$$\rho_Q(\mathfrak{p}^k) = \left(1 + \left\lfloor \frac{s}{2} \right\rfloor\right) N\mathfrak{p}^{k-1}\left(1 - \left(\frac{D_q}{\mathfrak{p}}\right)\right) + N\mathfrak{p}^k\left(1 - \frac{1}{N\mathfrak{p}}\right)\delta_{s \equiv 0 \text{ mod } 2}.$$

If $k \le r$,

$$\rho_Q(\mathfrak{p}^k) = N\mathfrak{p}^{k+\lfloor k/2 \rfloor}.$$

If $k > r, s = 0, r \equiv 0 \ mod \ 2$,

$$\rho_Q(\mathfrak{p}^k) = N\mathfrak{p}^{k+r/2-1}\left(N\mathfrak{p} - \left(\frac{D_q/\pi_\mathfrak{p}^r}{\mathfrak{p}}\right)\right).$$

If $k > r, s = 0, r \equiv 1 \ mod \ 2 \ and \ \left(\frac{\lambda}{\mathfrak{p}}\right) = \left(\frac{\alpha}{\mathfrak{p}}\right)$,

$$\rho_Q(\mathfrak{p}^k) = 2N\mathfrak{p}^{k+(r-1)/2}.$$

If $k > r, s = 0, r \equiv 1 \ mod \ 2 \ and \ \left(\frac{\lambda}{\mathfrak{p}}\right) = -\left(\frac{\alpha}{\mathfrak{p}}\right)$,

$$\rho_Q(\mathfrak{p}^k) = 0.$$

If $k > r, s > 0, k \le r + s$,

$$\rho_Q(\mathfrak{p}^k) = N\mathfrak{p}^{k+\lceil r/2 \rceil - 1}\left(\left(\left\lceil \frac{k-r}{2} \right\rceil - (r \ mod \ 2)\right)\left(1 - \left(\frac{D_q/\pi_\mathfrak{p}^r}{\mathfrak{p}}\right)\right) + N\mathfrak{p}^{(k-r)-2\lceil (k-r)/2 \rceil}\right).$$

If $k > r + s, s > 0$,

$$\rho_Q(\mathfrak{p}^k) = N\mathfrak{p}^{k+\lceil r/2 \rceil - 1}\left(\left(1 + \left\lfloor \frac{s}{2} \right\rfloor - (r \ mod \ 2)\right)\left(1 - \left(\frac{D_q/\pi_\mathfrak{p}^r}{\mathfrak{p}}\right)\right) + \left(1 - \frac{1}{N\mathfrak{p}}\right)\delta_{s \equiv 0 \ mod \ 2}\right).$$

Proof. This is proven exactly as in Appendix B of [Kha17] since in this case we have a diagonal representation of the same form as over $\mathbb{Q}$. The proof of the formulae for $r = 0$ depend only on the non-singularity of q over $\mathcal{O}_F/\mathfrak{p}$ not any sort of representation (and so they will continue to hold for non-singular forms over dyadic fields, as in the next section). □

Corollary B.1.3. *For any (odd) prime power* $\mathfrak{p}^k$,

$$\rho_Q(\mathfrak{p}^k) \le 3N\mathfrak{p}^{(3/2)k}.$$

Corollary B.1.4. *If* $r = 0$,

$$\widetilde{\rho}_Q(\mathfrak{p}^k) = \rho_Q(\mathfrak{p}^k).$$

If $r > 0, k \le r$,

$$\widetilde{\rho}_Q(\mathfrak{p}^k) = N\mathfrak{p}^{k+\lfloor k/2 \rfloor}\left(1 - \frac{1}{N\mathfrak{p}}\right)\delta_{k \equiv 0 \ mod \ 2}.$$

If $k > r > 0, s = 0$,

$$\widetilde{\rho}_Q(\mathfrak{p}^k) = \rho_Q(\mathfrak{p}^k)\left(1 - \frac{1}{N\mathfrak{p}}\right).$$

Proposition B.8 of [Kha17] also holds verbatim for this case (in particular note that that Proposition is only for primes not dividing 2) since we have the same diagonal representation.

B.2 Residue Characteristic 2

Recall F has mixed characteristic $(0,p)$. Suppose now that $p=2$.

Proposition B.2.1. *Suppose that $p=2$ and $q(\mathcal{O}_F^2)\mathcal{O}_F=\mathcal{O}_F$. Then either*

1. *q is equivalent to a form $ax^2+xy+by^2$, in which case $2\nmid D_q$; or*
2. *q is equivalent to a diagonal form ux^2+vy^2 where we may assume $u\in\mathcal{O}_F^\times$. In this case $D_q=-4uv$ is divisible by $\mathfrak{p}$.*

Proof. See the classification of [OMe63, §93]. The first case corresponds to a single factor in the Jordan splitting, and the second case corresponds to two 1-dimensional factors in the Jordan splitting. □

In the first case, the prime $\mathfrak{p}$ is regular, and so if $\mathfrak{p}\nmid\lambda$,

$$\rho_Q(\mathfrak{p}^k)=\widetilde{\rho}_Q(\mathfrak{p}^k)=N\mathfrak{p}^{k-1}\left(N\mathfrak{p}-\left(\frac{D_q}{\mathfrak{p}}\right)\right),$$

and the formulae of Proposition B.1.2 for $r=0$ continue to hold. Therefore

$$\rho_Q(\mathfrak{p}^k)\leq 5N\mathfrak{p}^{(3/2)k}.$$

In the diagonal case, let ρ_Q^1 be the count of solutions not congruent to $(0,0)$ modulo $\mathfrak{p}$. We see that if $\mathfrak{p}\nmid v$, the solutions not congruent to $(0,0)$ modulo $\mathfrak{p}$ lift by Hensel's Lemma for $k>2v_\mathfrak{p}(2)$, so

$$\rho_Q^1(\mathfrak{p}^{k+1})=N\mathfrak{p}\rho_Q^1(\mathfrak{p}^k)$$

By using the trivial bound $\rho_Q^1(\mathfrak{p}^{2v_\mathfrak{p}(2)+1})\leq N\mathfrak{p}^{2(2v_\mathfrak{p}(2)+1)}$, we get that in general (when $\mathfrak{p}\nmid v$),

$$\rho_Q^1(\mathfrak{p}^k)\leq N\mathfrak{p}^{2v_\mathfrak{p}(2)+1}N\mathfrak{p}^k=\left(2^{2[F:\mathbb{Q}_2]}N\mathfrak{p}\right)N\mathfrak{p}^k.$$

Now, the same recurrence relations as the proof of Lemmas B.4 and B.5 of [Kha17] proves that

$$\rho_Q(\mathfrak{p}^k)\leq N\mathfrak{p}^{k+\lceil r/2\rceil}\left(\min\left(\left\lceil\frac{k-r}{2}\right\rceil,1+\left\lfloor\frac{s}{2}\right\rfloor\right)2^{2[F:\mathbb{Q}_2]}N\mathfrak{p}+1\right).$$

Therefore,

$$\rho_Q(\mathfrak{p}^k)\leq 2^{2[F:\mathbb{Q}_2]+2}N\mathfrak{p}^{(3/2)k},$$

as required.

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